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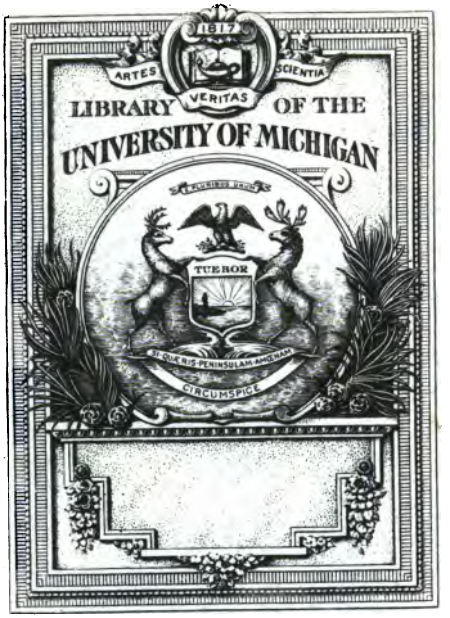

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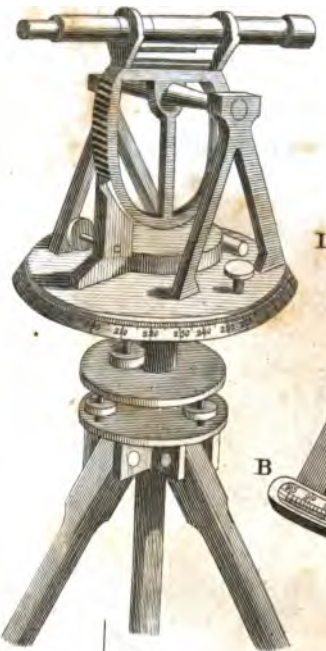

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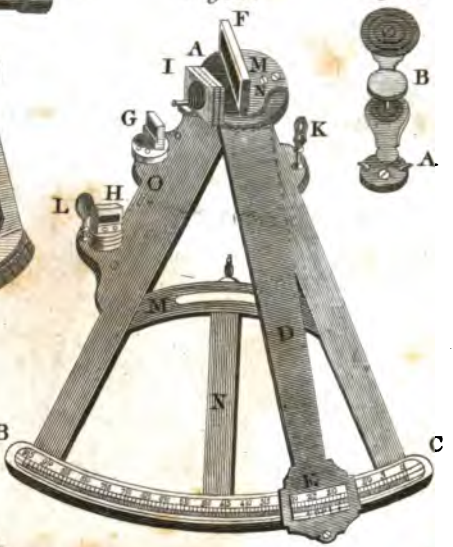

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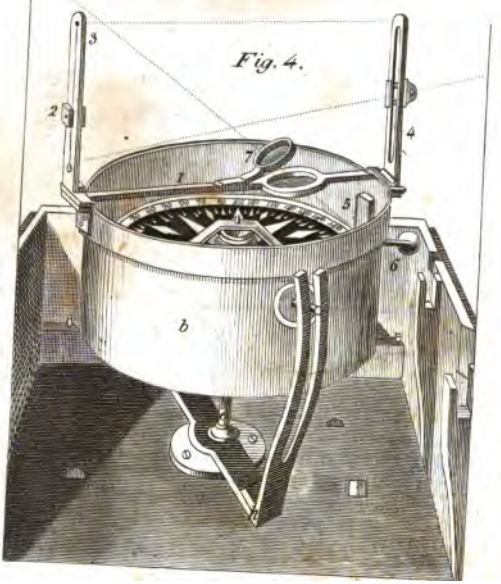

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PREFACE.

THE word Geometry imports no more than *to measure the earth*, or to measure the land; yet, in a larger and more proper sense, it is applied to all sorts of dimensions. It is generally supposed to have had its rise among the *Egyptians*, from the river *Nile's* destroying and confounding all their landmarks by its annual inundations, which laid them under the necessity of inventing certain methods and measures to enable them to distinguish and adjust the limits of their respective grounds when the waters were withdrawn. And this opinion is not entirely to be rejected, when we consider that *Moses* is said to have acquired this art when he resided at the *Egyptian* court. And *Achilles Tatius*, in the beginning of his introduction to *Aratus's Phænomena*, informs us that the *Egyptians* were the first who measured the heavens and the earth, and of course the earth first; and that their science in this matter was engraven on columns, and by that means delivered to posterity.

It is a matter of some wonder, that though *Surveying* appears to have been the first, or at least one of the first, of the mathematical sciences, the rest have met with much greater improvements from the pens of the most eminent mathematicians, while this seems to have been neglected; insomuch that I have not been able to meet with one author who has sufficiently explained the whole art in its theory and practice. For the most part, it has been treated of in a practical manner only; and the few who have undertaken the theory have in a great measure omitted the practice.

These considerations induced me to attempt a methodical, easy, and clear course of *Surveying*: how far I have succeeded in it must be determined by the impartial reader. The steps I have taken to render the whole evident and familiar are as follow:—

In section the first (Part the First) you have Decimal Fractions. The second section contains Involution and Evolution. The third section contains the nature and power of Logarithms, with their application, and the method of computing them.

The fourth section contains geometrical definitions, theorems, and problems, with the description and use of the sector, Gunter's scale, and other mathematical drawing instruments used by surveyors.

The fifth section contains Plane Trigonometry, right-angled and oblique, with a variety of rules and practical examples.

The first section (Part the Second) gives an account of the chains and measures used in *Great Britain* and *Ireland*, methods of surveying and of taking inaccessible distances by the chain only, with some necessary problems; also a particular description of the several instruments used in surveying, with their respective uses.

The second section contains the mensuration of heights and distances, with a great variety of problems and practical examples.

The third section contains the mensuration of areas, or the various methods of calculating the superficial contents of any field; also several new rules and problems, with practical examples, and various methods of finding the areas of maps from their geometrical construction; two of which, more concise than the rest, were first published in this work. Also, it contains four new and much more concise methods of determining the areas of surveys from the field-notes, or by calculation, than any hitherto published; to these is added the method of calculating the area of a survey, by having the meridian pass through the east or west point of the survey, with the method of discovering these points from the field-notes, and the method of correcting the errors by the pen, when the survey does not close: also another new method for calculating the area, by having a parallel of latitude pass through the north or south point of the survey. The whole geometrically considered and demonstrated.*

* The remaining part of the *Author's Preface* I have altered according to the arrangement and improvement of this *new edition*.—EDITOR.

The fourth section contains the nature of offsets, and the method of casting them up by the pen.

The fifth section contains the method of finding the areas by intersections.

The sixth section shows how to enlarge or diminish a map, or to reduce a map from one scale to another; also the manner of uniting separate maps of lands which join each other into one map of any assigned size.

The seventh section contains the method of dividing land, or of taking off or enclosing any given quantity.

Section the eighth treats of surveying harbours, shoals, sands, &c.

Section the ninth treats of levelling, adapted to the surveying of roads and hilly ground, with promiscuous questions.

Section the first (Part the Third) contains the astronomical methods of finding the latitude, variation of the compass, &c., with a description of the instruments used in these operations.

Section the second contains a description of the instruments requisite in astronomical observations.

Section the third shows how to find the variation of the compass; with a description of the azimuth compass, and its use.

In this edition is introduced a new set of accurate Mathematical Tables.

Truth calls upon me to acknowledge, that the methods of calculation herein set forth got their rise from those of the late *Thomas Burgh, Esq.*,* who first discovered a universal method for determining the areas of right-lined figures, and for which he obtained a reward of twenty thousand pounds sterling from the Irish Parliament. I hope, therefore, it cannot be construed as an intention in me to take from his great merit when I say, that the methods herein contained are much more concise and ready than his.

* This method, with very little alteration and improvement, in this country, is usually called the Pennsylvania Method of Calculation.—Ed.

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EXPLANATION

OF THE MATHEMATICAL CHARACTERS USED IN THIS WORK

$+$ signifies *plus*, or addition.
$-$ " *minus*, or subtraction.
$\times$ *or* . " multiplication.
$\div$ " division.
: :: : " proportion.
$=$ " equality.
$\surd$ " square root.
$\sqrt[3]{}$ " cube root, &c.
ϖ " difference between two numbers, when it is not known which is the greater.

Thus,

$5 + 3$, denotes that 3 is to be added to 5.
$6 - 2$, denotes that 2 is to be taken from 6.
7×3, or $7 . 3$, denotes that 7 is to be multiplied by 3.
$8 \div 4$, denotes that 8 is to be divided by 4.
$2 : 3 :: 4 : 6$, shows that 2 is to 3 as 4 is to 6.
$6 + 4 = 10$, shows that the sum of 6 and 4 is equal to 10.
$\surd 3$, or $3^{\frac{1}{2}}$, denotes the square root of the number 3.
$\sqrt[3]{5}$, or $5^{\frac{1}{3}}$, denotes the cube root of the number 5.
7^2, denotes that the number 7 is to be squared.
8^3, denotes that the number 8 is to be cubed.
Et cetera.

THE

THEORY AND PRACTICE

OF

SURVEYING.

The word Surveying, in the mathematics, signifies the art of measuring land, and of delineating its boundaries on a map.

The Surveyor, in the practice of this art, directs his attention, at first, to the tracing and measuring of lines; secondly, to the position of these lines in respect to each other, or the angles formed by them; thirdly, to the plan, or representation of the field or tract which he surveys; and fourthly, to the calculation of its area, or superficial content. When this art is employed in determining the variation of the compass, in observing and delineating coasts and harbours, their latitude, longitude, and soundings, together with the bearings of their most remarkable places from each other, it is usually denominated Maritime Surveying. This branch of Surveying, however, demands no other qualifications than those which should be thoroughly acquired by every land-surveyor who aspires to the character of an accomplished and skilful practitioner. Surveying, therefore, requires an intimate acquaintance with the several parts of the mathematics which are here inserted as an introduction to this treatise.

PART I.

Containing Decimal Fractions, Involution and Evolution, the Nature and Use of Logarithms, Geometry, and Plane Trigonometry.

SECTION I.

DECIMAL FRACTIONS.

If we suppose unity or any one thing to be divided into any assigned number of equal parts, this number is called the de-

nominator; and if we choose to take any number of such parts less than the whole, this is called the numerator of a fraction.

The numerator, in the vulgar form, is always written over the denominator, and these are separated by a small line thus $\frac{3}{4}$, or $\frac{5}{8}$; the first of these is called three-fourths, and the latter five-eighths, of an inch, yard, &c., or of whatever the whole thing originally consisted: the 4 and the 8 are the denominators, showing into how many equal parts the unit is divided; and the three and the five are the numerators, showing how many of those parts are under consideration.

Fractions are expressed in two forms, that is, either vulgarly or decimally.

All fractions whose denominators do not consist of a cipher or ciphers, set after unity, are called vulgar; and their denominators are always written under their numerators. The treatment of these, however, would be foreign to our present purpose. But fractions whose denominators consist of a unit prefixed to one or more ciphers, are called decimal fractions; the numerators of which are written without their denominators, and are distinguished from integers by a point prefixed; thus $\frac{2}{10}$, $\frac{42}{100}$, $\frac{172}{1000}$, in the decimal form, are expressed by .2, .42, .172.

The denominators of such fractions consisting always of a unit prefixed to as many ciphers as there are places of figures in the numerators, it follows, that any number of ciphers put after those numerators, will neither increase nor lessen their value: for $\frac{3}{10}$, $\frac{30}{100}$, and $\frac{300}{1000}$ are all of the same value, and will stand in the decimal form thus .3, .30, .300; but a cipher or ciphers prefixed to those numerators lessen their value in a tenfold proportion: for $\frac{3}{10}$, $\frac{03}{100}$, and $\frac{003}{1000}$, which in the decimal form we denote by .3, .03, and .003, are fractions, of which the first is ten times greater than the second; and the second, ten times greater than the third.

Hence it appears, that as the value and denomination of any figure, or number of figures, in common arithmetic is enlarged and becomes ten, or a hundred, or a thousand times greater, by placing one, or two, or three ciphers after it; so in decimal arithmetic, the value of any figure, or number of figures, decreases and becomes ten, or a hundred, or a thousand times less, while the denomination of it increases, and becomes so many times greater, by prefixing one, or two, or three ciphers to it: and that any number of ciphers before an integer, or after a decimal fraction, has no effect in changing its value.

SCALE OF NOTATION.

Integers.						Decimals.					
7 millions.	3 hundreds of thousands.	4 tens of thousands.	2 thousands.	1 hundreds.	8 tens.	6 units.	8 tenth parts.	7 hundredth parts.	5 thousandth parts.	3 ten thousandth parts.	2 hundred thousandth parts.	6 millionth parts.

ADDITION OF DECIMALS.

Write the numbers under each other according to the value or denomination of their places; which position will bring all the decimal points into a column, or vertical line, by themselves. Then, beginning at the right-hand column of figures, add in the same manner as in whole numbers, and put the decimal point in the sum directly beneath the other points.

EXAMPLES.

Add 4.7832, 3.2543, 7.8251, 6.03, 2.857, and 3.251 together. Place them thus,

4.7832
3.2543
7.8251
6.03
2.857
3.251

Sum = 28.0006

Add 6.2, 121.306, .75, 2.7, and .0007 together.

121.306
.75
2.7
.0007

Sum = 130.9567

What is the sum of 6.57, 1.026, .75, 146.5, 8.7, 526., 3.97, and .0271?

Answer, 693.5431.

What is the sum of 4.51, 146.071, .507, .0006, 132., 62.71, .507, 7.9, and .10712?

Answer, 354.31272.

SUBTRACTION OF DECIMALS.

Write the figures of the subtrahend beneath those of the minuend according to the denomination of their places, as directed in the rule of addition; then, beginning at the right-hand, subtract as in whole numbers, and place the decimal point in the difference exactly under the other two points.

EXAMPLES.

From 38.765 take 25.3741

25.3741

Difference = 13.3909

From 2.4 take .8472

.8472

Diff. = 1.5528

From 71.45 take 8.4837248.

Difference = 62.9662752.

From 84 take 82.3412.

Diff. = 1.6588.

MULTIPLICATION OF DECIMALS.

Set the multiplier under the multiplicand without any regard to the situation of the decimal point; and having multiplied as in whole numbers, cut off as many places for decimals in the product, counting from the right-hand towards the left, as there are in both the multiplicand and multiplier: but if there be not a sufficient number of places in the product, the defect may be supplied by prefixing ciphers thereto.

For the denominator of the product being a unit, prefixed to as many ciphers as the denominators of the multiplier and multiplicand contain of ciphers, it follows that the places of decimals in the product will be as many as in the numbers from whence it arose.

EXAMPLES.

Multiply 48.765 by .003609.

.003609

438885
292590
146295

Product = .175992885

Multiply .121
by .14

484
121

Product = .01694

Multiply 121.6 by 2.76

2.76

7296
8512
2432

Product = 335.616

Multiply .0089789 by 1085.
Product = 9.7421065.
Multiply .248723 by .13587.
Product = .03379399401.

DIVISION OF DECIMALS.

Divide as in whole numbers; observing that the divisor and quotient together must contain as many decimal places as there are in the dividend. If, therefore, the dividend have just as many places of decimals as the divisor has, the quotient will be a whole number without any decimal figures. If there be more places of decimals in the dividend than there are in the divisor, point off as many figures in the quotient for decimals, as the decimal places in the dividend exceed those in the divisor; the want of places in the quotient being supplied by prefixing ciphers. But if there be more decimal places in the divisor than in the dividend, annex ciphers to the dividend, so that the decimal places here may be equal in number to those in the divisor; and then the quotient will be a whole number, without fractions.

When there is a remainder, after the division has been thus performed, annex ciphers to this remainder, and continue the operation till nothing remains, or till a sufficient number of decimals shall be found in the quotient.

EXAMPLES.

Divide .144 by .12.

```
.12).144(1.2 = quotient.
     12
    ---
     24
     24
    ---
      0
```

Divide 63.72413456922 by 2718.

```
2718)63.72413456922(.02344522979 = quotient.
     5436
    ------
     9364
     8154
    ------
     12101
     10872
    ------
     12293
     10872
    ------
     14214
     13590
    ------
      6245
      5436
    ------
      8096
      5436
    ------
     26609
     24462
    ------
     21472
     19026
    ------
      24462
      24462
    ------
          0
```

There being 11 decimal figures in the dividend, and none in the divisor, 11 figures are to be cut off in the quotient; but as the quotient itself consists of but 10 figures, prefix to them a cipher to complete that number.

Divide 1.728 by .012

```
.012)1.728(144 = quotient.—
    12
   ---
    52
    48
   ---
     48
     48
    ---
      0
```

Because the number of decimal figures in the divisor and dividend are alike, the quotient will be integers.

Divide 2 by 3.1416

```
3.1416)2.0000,0(0.636618+ = quotient.
     1 8849 6
     ---------
      115040
       94248
      -------
      207920
      188496
      ------
       194240
       188496
      -------
        57440
        31416
       ------
       260240
       251328
       ------
         8912+
```

In this example there are four decimal figures in the divisor, and none in the dividend; therefore, according to the rule, four ciphers are annexed to the dividend, which, in this condition, is yet less than the divisor. A cipher must then be put in the quotient in the place of integers, and other ciphers annexed to the dividend; and the division being now performed, the decimal figures of the quotient are obtained.

Divide 7234.5 by 6.5 Quotient = 1113.
Divide 476.520 by .423 ——— = 1126.5+
Divide .45695 by 12.5 ——— = .0365+
Divide 2.3 by 96 ——— = .02395+
Divide 87446071 by .004387 — = 19933000000.
Divide .624672 by 482 ——— = .001296.

REDUCTION OF DECIMALS.

RULE I.

To reduce a Vulgar Fraction to a Decimal of the same value.

Having annexed a sufficient number of ciphers, as decimals, to the numerator of the vulgar fractions, divide by the denominator; and the quotient thence arising will be the decimal fraction required.

EXAMPLE.

Reduce $\frac{3}{4}$ to a decimal fraction.

4)3.00

.75=decimal required.

For $\frac{3}{4}$ of one acre, mile, yard, or any thing, is equal to $\frac{1}{4}$ of 3 acres, miles, yards, &c.; therefore if 3 be divided by 4, the quotient is the answer required.

Reduce $\frac{4}{5}$ to a decimal fraction. Answer .4
Reduce $\frac{12}{25}$ - - - - - - - .48
Reduce $\frac{235}{2048}$ - - - - - - - .1146789
Reduce $\frac{7}{9}$ - - - - - - - .7777+
Reduce $\frac{21}{23}$ - - - - - - - .9130434+

Reduce $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, and so on to $\frac{1}{30}$, to their corresponding decimal fractions; and in this operation the various modes of interminate decimals may be easily observed.

RULE II.

To reduce Quantities of the same, or of different Denominations, to Decimal Fractions of higher Denominations. —

If the given quantity consist of one denomination only, write it as the numerator of a vulgar fraction; then consider how many of this make one of the higher denomination, mentioned in the question, and write this latter number under the former, as the denominator of a vulgar fraction. When this has been done, divide the numerator by the denominator, as directed in the foregoing rule, and the quotient resulting will be the decimal fraction required.

But if the given quantity contain several denominations, reduce them to the lowest term for the numerator; reduce likewise that quantity whose fraction is sought to the same denomination, for the denominator of a vulgar fraction; then divide as before directed.

EXAMPLES.

Reduce 9 inches to the decimal of a foot.

The foot being equal to 12 inches, the vulgar fraction will be $\frac{9}{12}$; then 12)9.00

.75=decimal fraction required.

Reduce 8 inches to the decimal of a yard.

```
                    8 inches
                    ----
1 yard × 3 × 12 = 36
                36)8.0(.22+ = Answer.
                   7 2
                   ---
                    80
                    72
                    --
                     8
```

Reduce 5 furlongs 00 perches to the decimal of a mile.

```
1 mile              5 furlongs
8                   40
---                 ---
8 fur.              200
40                  --- = vulgar fraction.
---                 320
320 per.
                320)200.0(.625 = decimal sought.
                    192 0
                    -----
                      800
                      640
                      ---
                      1600
                      1600
```

Reduce 21 minutes 54 seconds to the decimal of a degree. Ans. .365.

Reduce .056 of a pole to the decimal of an acre. Ans. .00035.

Reduce 13 cents to the decimal of an eagle. Ans. .013.

Reduce 14 minutes to the decimal of a day. Ans. .00972+

Reduce 3 hours 46 minutes to the decimal of a week. Ans. 0224206+

RULE III.

To find the value of Decimal Fractions in terms of the lower denominations.

Multiply the given decimal by the number of the next lower denomination which makes an integer of the present, and point off as many places at the right-hand of the product, for a remainder, as there are figures in the given decimal. Multiply this remainder by the number of the next inferior denomination, and point off a remainder as before. Proceed in this manner through all the parts of the integer, and the several denominations standing on the left-hand are the value required.

EXAMPLES.

Required the value of .3375 of an acre.

4 = number of roods in an acre.

1.3500

40 = number of perches in a rood.

14.0000

The value, therefore, is 1 rood 14 perches.

What is the value of .6875 of a yard?

3 = number of feet in a yard.

2.0625

12 = number of inches in a foot.

.7500

12 = number of lines in an inch.

9.0000

The answer here is 2 feet 9 lines.

What is the value of .084 of a furlong? Ans. 3 per. 1 yd. 2 ft. 11 in.

What is the value of .683 of a degree? Ans. 40 m. 58 sec. 48 thirds.

What is the value of .0053 of a mile? Ans. 1 per. 3 yds. 2 ft. 5 in.+

What is the value of .036 of a day? Ans. 51′ 50″ 24‴

PROPORTION IN DECIMAL FRACTIONS.

Having reduced all the fractional parts in the given quantities to their corresponding decimals, and having stated the three known terms, so that the fourth, or required quantity, may be as much greater or less than the third as the second term is greater or less than the first, then multiply the second and third terms together, and divide the product by the first term, and the quotient will be the answer;—in the same denomination with the third term.

EXAMPLES.

If 3 acres 3 roods of land can be purchased for 93 dollars 60 cents, how much will 15 acres 1 rood cost at that rate?

3 acs. 3 rds. = 3.75 acres.
15 acs. 1 rd. = 15.25 acres.
$93, 60 cents = $93.60

Then 3.75 : 15.25 : : 93.60 :

15.25

46800
18720
46800
9360

3.75)1427.4000($380.64 = Answer.
1125

3024
3000

2400
2250

1500
1500

If a clock gain 14 seconds in 5 days 6 hours, how much will it gain in 17 days 15 hours? Ans. 47 seconds.

If 167 dollars 85 cents gain 12 dollars 33 cents interest in a year, at what rate per cent. is this interest? Ans. 6.56+

SECTION II.

INVOLUTION AND EVOLUTION.

INVOLUTION is the method of raising any number, considered as the root, to any required power.

Any number, whether given or assumed at pleasure, may be called the root or first power of this number; and its other powers are the products that result from multiplying the number by itself, and the last product by the same number again, and so on to any number of multiplications.

The index, or exponent, is the number denoting the height, or degree of the power, being always greater by one than the number of multiplications employed in producing the power. It is usually written above the root, as in the following EXAMPLE, where the method of involution is plainly exhibited.

Required the fifth power of 8 = the root, or first power.
first multiply by - - 8

then multiply the product 64 = 8^2 = square, or second power.
by 8

&c. 512 = 8^3 = cube, or third power.
8

4096 = 8^4 = biquadrate, or fourth power.
8

32768 = 8^5 = Answer.

EXAMPLES FOR EXERCISE.

What is the second power of 3.05? Ans. 9.3025.
What is the third power of 85.3? Ans. 620650.477.
What is the fourth power of .073? Ans. .000028398241.
What is the eighth power of .09? Ans. .00.00.00.0043046721.

Note.—When two or more powers are multiplied together, their product is that power whose index is the sum of the indices of the factors, or powers multiplied.

EVOLUTION is the method of extracting any required root from any given power.

Any number may be considered as a power of some other number; and the required root of any given power is that

number which being multiplied into itself a particular number of times produces the given power; thus if 81 be the given number, or power, its square or second root is 9; because $9 \times 9 = 9^2 = 81$; and 3 is its biquadrate, or fourth root, because $3 \times 3 \times 3 \times 3 = 3^4 = 81$. Again, if 729 be the given power, and its cube root be required, the answer is 9, for $9 \times 9 \times 9 = 729$; and if the sixth root of that number be required, it is found to be 3, for $3 \times 3 \times 3 \times 3 \times 3 \times 3 = 729$.

The required power of any given number, or root, can always be obtained exactly, by multiplying the number continually into itself; but there are many numbers from which a proposed root can never be completely extracted;—yet by approximating with decimals, these roots may be found as exact as necessity requires. The roots that are found complete are denominated *rational* roots, and those which cannot be found completed, or which only approximate, are called *surd*, or irrational roots.

Roots are usually represented by these characters or exponents:

$\sqrt{}$, or $\frac{1}{2}$ which signifies the square root; thus, $\sqrt{9}$, or $9^{\frac{1}{2}} = 3$.

$\sqrt[3]{}$ or $\frac{1}{3}$ cube root; $\sqrt[3]{64}$, or $64^{\frac{1}{3}} = 4$.

$\sqrt[4]{}$ or $\frac{1}{4}$ biquadrate root; $\sqrt[4]{16}$, or $16^{\frac{1}{4}} = 2$, &c.

Likewise $8^{\frac{3}{2}}$ signifies the square root of 8 cubed; and, in general, the fractional indices imply that the given numbers are to be raised to such powers as are denoted by their numerators, and that such roots are to be extracted from these powers as are denoted by their denominators.

RULE

For extracting the Square Root.

Commencing at the unit figure, cut off periods of two figures each, till all the figures of the given number are exhausted.*

The first figure of the required root will be the square root

* In dividing a decimal, or a number consisting of a whole number with a decimal, into periods, the division must also commence at the unit figure or decimal point, and must be continued both ways, if there be a whole number; and if there be an odd figure at the end of the decimal, a cipher, or if it be a periodical decimal, the figure that would next arise, from its continuation, must be annexed; thus 417.245 will be divided thus, 4'17'.24'50: 41.66666, &c. thus, 41'.66'66'66: and .567 thus, 56'70, &c.

See the Editor's "*Elementary Treatise on Arithmetic, in Theory and Practice*," page 219.—Ed.

of the first period, or of the greatest square root contained in it, if it be not a square itself.

Subtract the square of this figure from the first period; to the remainder annex the next period for a dividend; and for part of a divisor, double the part of the root already obtained.

Try how often this part of the divisor is contained in the dividend wanting the last figure, and annex the figures thus found to the parts of the root and of the divisor already determined.

Thus multiply and subtract as in division; to the remainder bring down the next period, and, adding to the divisor the figure of the root last found, proceed as before.*

If any thing remain after continuing the process till all the figures in the given number have been used, proceed in the same manner to find decimals, adding, to find each figure, two ciphers, or if the given number end in an interminate decimal, the two figures that would next arise from its continuation.

To extract the root of a fraction, reduce it to its simplest form, if it be not so already, and extract the root of both terms, if they be complete powers: otherwise divide the root of their product by the denominator.

The root may also be found by reducing the fraction to a decimal, if it be not such already, and taking the root of the decimal.

EXAMPLES.

Required the square root of 1710864.

	1'71'08'64'
1	1710864(1308 = Answer.
1	1
23	71
3	69
2608	20864
	20864

* *The principle* on which the preceding rule depends is, *that the square of the sum of two numbers is equal to the squares of the numbers with twice their product.* Thus, the square of 34 is equal to the squares of 30 and of 4 with twice the product of 30 and 4; that is, to $900+2\times30\times4+16=1156$. Here, in extracting the second root of 1156, we separate it into two parts, 1100 and 56. Thus 1100 contains 900, the square of 30, with the remainder 200; the first part of the root is therefore 30, and the remainder $200+56$, or 256. Now, according to the principle above men-

Required the square root of 16007.3104.

	1'60'07'.31'04'
1	16007.3104(126.52 = Answer.
1	1
22	60
2	44
246	1607
6	1476
2525	13131
5	12625
25302	50604
	50604

EXAMPLES FOR EXERCISE.

Required the square root of 298116. Ans. 546.
Required the square root of 348.17320836. Ans. 18.6594.
Required the square root of 17.3056. Ans. 4.16.
Required the square root of .000729. Ans. .027.
Required the square root of $17\frac{3}{8}$. Ans. 4.168333+

TO EXTRACT THE CUBE ROOT.

RULE I.—Commencing at the unit figure, cut off periods of three figures each, till all the figures of the given number are exhausted. Then find the greatest cube number contained in the *first* period, and place the cube root of it in the quotient. Subtract its cube from the first period, and bring down the next three figures; divide the number thus brought down by 300 times the square of the first figure of the root, and it will give the second figure; add 300 times the square of the first figure, 30 times the product of the first and second figures, and the square of the second figure together, for a divisor; then

tioned, this remainder must be twice the product of 30, and the part of the root still to be found, together with the square of that part. Now, dividing 256 by 60, the double of 30, we find for quotient 4; then this part being added to 60, the sum is 64, which being multiplied by 4, the product 256 is evidently twice the product of 30 and 4, together with the square of 4. In the same manner the operation may be illustrated in every case. The rule, however, is best demonstrated by Algebra.

See my Treatise on this subject, page 231, second edition.—ED.

multiply this divisor by the second figure, and subtract the result from the dividend, and then bring down the next period, and so proceed till all the periods are brought down.*

To extract the cube root of a fraction, reduce it to a decimal, and then extract the root; or multiply the numerator by the square of the denominator, find the cube root of the product, and divide by the denominator.

The cube root of a mixed number is generally best found by reducing the fractional part to a decimal, if it be not so already, and then extracting the root. It may be also found by reducing the given number to an improper fraction, and then working according to the preceding directions.

EXAMPLES.

1. Required the cube root of 48228.544.

$3^2 \times 300 = 2700$	48'228'.544'(36.41 Root.
$3 \times 30 = 90$	27
Divisor 2790	21228 Resolvend.
	19656 Subtrahend.
$3^2 \times 300 \times 6 = 16200$	
$3 \times 30 \times 6^2 = 3240$	1572.544 Resolvend.
$6^3 = 216$	1572.544 Subtrahend.

Subtrahend 19656

$36^2 \times 300 = 388800$
$36 \times 30 = 1080$

Divisor 389880

$36^2 \times 300 \times 4 = 1555200$
$36 \times 30 \times 4^2 = 17280$
$4^3 = 64$

Subtrahend .1572544

Ex. 2. What is the cube root of 62570773? Ans. 397.
Ex. 3. What is the cube root of 51478848? Ans. 372.
Ex. 4. What is the cube root of 84.604519? Ans. 4.39.
Ex. 5. What is the cube root of 16974593? Ans. 257.

* The reason of this rule will appear evident from the following illustration. The cube of 25, for instance, is equivalent to the cube of 20 added to the cube of 5, together with the sum of $300 \times 4 \times 5 + 30 \times 2 \times 5 \times 5$, or, which is the same thing, 25 is equal to $20 + 5$, and therefore 25 cubed

2. *To extract the Cube Root by another Method.**

1. By trials find the nearest rational cube to the given number, whether it be greater or less, and call it the assumed cube.

2. Then say, by the Rule of Three, as the sum of the given number and double the assumed cube is to the sum of the assumed, and double the given number, so is the root of the assumed cube to the root required, nearly. Or, as the first sum is to the difference of the given and assumed cube, so is the assumed root to the difference of the roots, nearly.

3. By using, in like manner, the cube of the root last found as a new assumed cube, another root will be obtained still nearer. And so on as far as we please; using always the cube of the last found root for the assumed cube.

EXAMPLES.

1. To find the cube root of 21035.8.

Here the root is soon found between 27 and 28. Taking therefore 27, its cube is 19683, which is the assumed cube. Then,

19683	21035.8
2	2
39366	42071.6
21035.8	19683

As 60401.8 : 61754.6 : : 27 : 27.6047,

---

is equal to $20+5$ cubed; but $20+5$ cubed is equivalent to $8000+300\times4\times5+30\times2\times5\times5+125$, or to $20^3+(300\times4+30\times2\times5+5\times5)\times5=48228544$.

Thus, $\left.\begin{matrix}20+5\\20+5\end{matrix}\right\}$ Multiplied.

$$20\times20+5\times20$$
$$+5\times20+25$$

Multiplied, $\left\{\begin{matrix}20\times20+2\times5\times20+25=\text{second power.}\\20+5\end{matrix}\right.$

$$20\times20\times20+2\times5\times20\times20+20\times25$$
$$+5\times20\times20+2\times20\times25+125$$

$20\times20\times20+3\times5\times20\times20+3\times20\times25+125$=3d power
or, $8000+300\times4\times5+30\times2\times25+125$.

Here the rule is evident. In the same manner, the operation may be illustrated in every case. For a demonstration of this rule in general terms, the reader is referred to the Editor's "Treatise on Algebra, Theoretical and Practical."—Ed.

* This rule is found in Hutton's Mathematics. There have been different rules given for extracting the cube root, among which this, and another rule given in Pike's Arithmetic (by approximation), are very expeditious.

Therefore 27.6047 is the root nearly.

Again, by repeating the operation, and taking 27.6047 for the assumed root, it will give 27.60491 the root still nearer.

2. Required the cube root of 3214? Ans. 14.75758.

3. Required the cube root of 2? Ans. 1.25992.

4. Required the cube root of 256? Ans. 6.349.

SECTION III.

OF LOGARITHMS.

LOGARITHMS are a series of numbers, so contrived, that by them the work of multiplication may be performed by addition; and the operation of division may be done by subtraction. Or, —Logarithms are the indices, or series of numbers in arithmetical progression, corresponding to another series of numbers in geometrical progression. Thus,

0, 1, 2, 3, 4, 5, 6, &c. indices or logarithms.
1, 2, 4, 8, 16, 32, 64, &c. geometrical progression.

Or,

0, 1, 2, 3, 4, 5, 6, &c. ind. or log.
1, 3, 9, 27, 81, 243, 729, &c. geometrical series.

Or,

0*, 1, 2, 3, 4, 5, 6, &c. ind. or log.
1, 10, 100, 1000, 10000, 100000, 1000000, &c. geometrical series,—where the same indices serve equally for any geometrical series or progression.

Hence it appears that there may be as many kinds of indices, or logarithms, as there can be taken kinds of geometrical series. But the logarithms most convenient for common uses are those adapted to a geometrical series increasing in a tenfold progression, as in the last of the foregoing examples.

In the geometrical series 1, 10, 100, 1000, &c. if between the terms 1 and 10 the numbers 2, 3, 4, 5, 6, 7, 8, 9 were interposed, indices might also be adapted to them in an arith-

* In any system of logarithms the log. of 1 is 0; for logarithms may be considered as the exponents of the powers to which a given or invariable number must be raised, in order to produce all the common or natural numbers, therefore by assuming $x^0=a$, then by squaring $x^0=a^2$ hence $a^2=a$, and consequently by division $a=1$, from whence it is evident that the log. of 1 is always $= 0$, in any system; for more on this subject, and the algebraical form of the rule for computing logarithms, *see* Bonnycastle's Algebra, page 200, New-York edition; or my Treatise on Algebra, page 332, second edition.—ED.

metical progression, suited to the terms interposed between 1 and 10, considered as a geometrical progression. Moreover, proper indices may be found to all the numbers, that can be interposed between any two terms of the geometrical series.

But it is evident that all the indices to the numbers under 10, must be less than 1; that is, they must be fractions. Those to the numbers between 10 and 100, must fall between 1 and 2; that is, they are mixed numbers, consisting of one and some fraction. Likewise the indices to the numbers between 100 and 1000, will fall between 2 and 3; that is, they are mixed numbers, consisting of 2 and some fraction; and so of the other indices.

Hereafter the integral part only of these indices will be called the index; and the fractional part will be called the logarithm. The computation of these fractional parts is called *making logarithms;* and the most troublesome part of this work is to make the logarithms *of prime numbers*, or those which cannot be divided by any other numbers than themselves and unity.

RULE

*For computing the Logarithms of Numbers.**

Let the sum of its proposed number and the next less number be called A. Divide 0.8685889638+ by A, and reserve

* The number 0.8685889638+ is twice the reciprocal of the hyperbolic log. 2.302585093, which is the log. of 10, according to the first form of Lord Napier, the inventor of logarithms; which log. according to the excellent Sir I. Newton's method is calculated thus; let DFD (Pl. 14, *fig.* 1) be an hyperbola whose centre is C, vertex F, and interposed square CAFE=1. In CA take AB and Ab, on each side $= \frac{1}{10}$, or 0.1; and, erecting the perpendiculars BD, bd, half the sum of the spaces AD and Ad will be $=0.1+\frac{0.001}{3}+\frac{0.00001}{5}+\frac{0.0000001}{7}$, &c.
and the half diff. $=\frac{0.01}{2}+\frac{0.0001}{4}+\frac{0.000001}{6}+\frac{0.00000001}{8}$, &c.
Which reduced will stand thus,

0.1000000000000,	0.0050000000000 Sum of these=0.1053605156577=A*d*
3333333333	250000000 And the diff. =0.0953101798043=AD
20000000	1666666 In like manner putting AB and A*b*
142857	12500 each = 0.2 there is obtained
1111	100 A*d* = 0.2231435513142, and
9	1 AD = 0.1823215567939.
0.1003353477310,	0.0050251679267

Having thus the hyperbolic logarithms of the four decimal numbers 0.8, 0.9, 1.1, and 1.2; and since $\frac{1.2}{0.6}\times\frac{1.2}{0.9}=2$, and 0.8 and 0.9 are less than unity, adding their logarithms to double the log. of 1.2, we have 0.6931471805597, the hyperbolic log. of 2. To the triple of this adding the log. of 0.8, because $\frac{2\times2\times2}{0.8}=10$, we have 2.3025850929933, the log. of 10. Hence by one addition

the quotient. Divide the reserved quotient by the square of A, and reserve this quotient. Divide the last reserved quotient by the square of A, reserving the quotient still; and thus proceed as long as division can be made. Write the reserved quotients orderly under one another, the first being uppermost. Divide these quotients respectively by the odd numbers 1, 3, 5, 7, 9, 11, &c.; that is, divide the first reserved quotient by 1, the second by 3, the third by 5, the fourth by 7, &c., and let these quotients be written orderly under one another; add them together, and their sum will be a logarithm. To this logarithm add the logarithm of the next less number, and the sum will be the logarithm of the number proposed.

EXAMPLE 1.

Required the logarithm of the number 2.

Here the next less number is 1, and $2+1=3=A$, and A^2 or $3^2=9$; then

3)0.868588964

9)0.289529654 ÷ 1 = 0.289529654

9)0.032169962 ÷ 3 = 0.010723321

9)0.003574440 ÷ 5 = 0.000714888

9)0.000397160 ÷ 7 = 0.000056737

9)0.000044129 ÷ 9 = 0.000004903

9)0.000004903 ÷ 11 = 0.000000446

9)0.000000545 ÷ 13 = 0.000000042

0.000000061 ÷ 15 = 0.000000004

are found the logarithms of 9 and 11: And thus the logarithms of all the prime numbers are prepared, that is, 2, 3, 5, 11, &c.

Moreover, by only depressing the numbers above computed, lower in the decimal places, and adding, are obtained the logarithms of the decimals 0.98, 0.99, 1.01, 1.02; as also of these, 0.998, 0.999, 1.001, 1.002. And hence, by addition and subtraction, will arise the logarithms of the primes 7, 13, 17, 37, &c. All which logarithms being divided by 2.3025850929933 (the hyperbolic log. of 10), or multiplied by its reciprocal, .4342944819, give the common logarithms to be inserted in the table.

Note.—For further illustration on this subject, the reader is referred to Hutton's Tables.

To this logarithm 0.301029995
add the logarithm of 1=0.000000000

Their sum=0.301029995=log. of 2.

The manner in which the division is here carried on may be readily perceived by dividing, in the first place, the given decimal by A, and the succeeding quotients by A^2; then letting these quotients remain in their situation, as seen in the example, divide them respectively by the odd numbers, and place the new quotients in a column by themselves. By employing this process, the operation is considerably abbreviated.

EXAMPLE 2.

Required the logarithm of the number 3.
Here the next less number is 2; and 3+2=5=A, and A^2=25.

5)0.868588964

25)0.173717793÷ 1=0.173717793

25)0.006948712÷ 3=0.002316237

25)0.000277948÷ 5=0.000055590

25)0.000011118÷ 7=0.000001588

25)0.000000445÷ 9=0.000000049

0.000000018÷11=0.000000002

To this logarithm 0.176091259
add the logarithm of 2=0.301029995

Their sum=0.477121254=log. of 3.

Then, because the sum of the logarithms of numbers gives the logarithm of their product; and the difference of the logarithms gives the logarithm of the quotient of the numbers: from the two preceding logarithms, and the logarithm of 10, which is 1, a great many logarithms can be easily made, as in the following examples.

Example 3. Required the logarithm of 4.
Since 4=2×2, then to the logarithm of 2=0.301029995
add the logarithm of 2=0.301029995

The sum=logarithm of 4=0.602059990

Example 4. Required the logarithm of 5.
$10 \div 2$ being $= 5$, therefore from the logarithm of
$10 = 1.000000000$
subtract the logarithm of $2 = 0.301029995$

the remainder is the logarithm of $5 = 0.698970005$

Example 5. Required the logarithm of 6.
$6 = 3 \times 2$, therefore to the logarithm of $3 = 0.477121254$
add the logarithm of $2 = 0.301029995$

their sum = logarithm of $6 = 0.778151249$

Example 6. Required the logarithm of 8.
$8 = 2^3$, therefore multiply the logarithm of $2 = 0.301029995$
by 3

The product = logarithm of $8 = 0.903089985$

Example 7. Required the logarithm of 9.
$9 = 3^2$, therefore the logarithm of $3 = 0.477121254$
being multiplied by 2

the product = logarithm of $9 = 0.954242508$

Example 8. Required the logarithm of 7.
Here the next less number is 6, and $7+6=13=A$, and $A^2=169$.

13)0.868588964

169)0.066814536 ÷ 1 = 0.066814536

169)0.000395352 ÷ 3 = 0.000131784

169)0.000002339 ÷ 5 = 0.000000468

0.000000014 ÷ 7 = 0.000000002

To this logarithm = 0.066946790
add the logarithm of $6 = 0.778151249$

Their sum = 0.845098039 = logarithm of 7.

The logarithm	of 12		of 3 and 4
	of 14		of 7 and 2.
	of 15	is equal to the sum of the	of 3 and 5.
	of 16	logarithms	of 4 and 4.
	of 18		of 3 and 6.
	of 20		of 4 and 5.

The logarithms of the prime numbers 11, 13, 17, 19, &c. being computed by the foregoing general rule, the logarithms of the intermediate numbers are easily found by composition and division. It may however be observed, that the operation is shorter in the larger prime numbers; for when any given number exceeds 400, the first quotient being added to the logarithm of its next lesser number, will give the logarithm sought, true to eight or nine places; and therefore it will be very easy to examine any suspected logarithm in the Tables.

For the arrangement of logarithms in a table, the method of finding the logarithm of any natural number, and of finding the natural number corresponding to any given logarithm therein,—likewise for particular rules concerning the indices, the reader will consult Table 1, *with its explanation at the end of this treatise.*

MULTIPLICATION.

Two or more numbers being given, to find their product by Logarithms.

RULE.

Having found the logarithms of the given numbers in the table, add them together, and their sum is the logarithm of the product; which logarithm being found in the table, will give a natural number, that is, the product required.

Whatever is carried from the decimal part of the logarithm is to be added to the affirmative indices, but subtracted from the negative. Likewise the indices must be added together when they are all of the same kind, that is, when they are all affirmative, or all negative; but when they are of different kinds, the difference must be found, which will be of the same denomination with the greater.

Example 1. Required the product of 86.25 multiplied by 6.48.

Log. of 86.25 = 1.935759
Log. of 6.48 = 0.811575

Product = 558.9 = 2.747334*

* For the method of finding the natural number, answering to the sum of the logarithms, the reader will consult Table 1, at the end of this treatise.

Example 2. Required the product of 46.75 and .3275.

Log. of 46.75 = 1.669782
Log. of .3275 = —1.515211

Product = 15.31 + = 1.184993

Here the +1 that is to be carried from the decimals cancels the —1, and consequently there remains 1 in the upper line to be set down.

Example 3. Required the product of 3.768, 2.053, and .007693.

Log. of 3.768 = 0.576111
Log. of 2.053 = 0.312389
Log. of .007693 = —3.886096

Product = .05951 + = —2.774596

In this example there is 1 to carry from the decimal part of the logarithms, which, subtracted from —3, the negative index, leaves 2, the index of the sum of the logarithms, and is negative.

Example 4. Required the product of 27.63, 1.859, .7258, and 0.3591.

Log. of 27.63 = 1.441381
Log. of 1.859 = 0.269279
Log. of .7258 = —1.860817
Log. of .03591 = —2.555215

Product nearly = 1.339 = 0.126692

In this example there is 2 to carry from the decimal part of the logarithms, which added to 1, the affirmative index, makes 3, from this take 3, the sum of the negative indices, the remainder is 0, which is the index of the sum of the logarithms.

5. Required the product of 23.14 and 50.62, by logarithms. Ans. 117.1347

6. Required the product of 3.12567, .02868, and .12379, by logarithms. Ans. .01109705

7. Required the product of .1508, .0139, and 756.9, by logarithms. Ans. 1.586553

8. Required the product of 637.8 and 89.27, by logarithms. Ans. 56936.406

9. Required the product of 14 and 8.45, by logarithms. Ans. 118.30

DIVISION.

Two numbers being given, to find how many times one is contained in the other by Logarithms.

RULE.

From the logarithm of the dividend subtract the logarithm of the divisor, and the remainder will be the logarithm whose corresponding natural number will be the quotient required.

In this operation, the index of the divisor must be changed from affirmative to negative, or from negative to affirmative; and then the difference of the affirmative and negative indices must be taken for the index to the logarithm of the quotient. Likewise when 1 has been borrowed in the left-hand place of the decimal part of the logarithm, add it to the index of the divisor, if affirmative; but subtract it, if negative; and let the index thence arising be changed and worked with as before.

Example 1. Divide 558.9 by 6.48.

Log. of 558.9 = 2.747334
Log. of 6.48 = 0.811575

Quotient = 85.25 = 1.935759

Here, the 1 to be taken from the decimals is taken as —1, which when added to 2, the index of the dividend, leaves 1 for the index of the quotient; that is, 2—1=1.

Example 2. Divide 15.31 by 46.75.

Log. of 15.31 = 1.184975
Log. of 46.75 = 1.669782

Quotient = .3275 = —1.515193

Here, the 1 to be taken from the decimals is added to 1, the index of the divisor makes 2; this with its sign changed is —2, from which subtracting 1, the index of the dividend, the remainder is —1, which is negative, because the negative index is greater.

Example 3. Divide .05951 by .007693.

Log. of .05951 = —2.774590
Log. of .007693 = —3.886096

Quotient = 7.735 = 0.888494

Here, the 1 to be taken from the decimals is subtracted from —3, which leave —2, this changed is +2; and this added to —2, the other index, gives 2—2=0.

Example 4. Divide .6651 by 22.5.

Log. of .6651 = —1.822887
Log. of 22.5 = 1.352183

Quotient = .02956 = —2.470704

Here, +1 in the lower index, is changed into —1, and this added to —1, the other index, gives —1—1, or —2, the index of the result.

5. Required the quotient of 125 divided by 1728, by logarithms. Ans. .0723379

6. Divide 1728.95 by 1.10678, by logarithms. Ans. 1562.144

7. Divide .067859 by 1234.59, by logarithms. Ans. .0000549648

8. Divide .7438 by 12.9470, by logarithms. Ans. .057449

9. Divide .06314 by .007241, by logarithms. Ans. 8.71979

PROPORTION,

Or the Rule of Proportion in Logarithms.

RULE.

Having stated the three given terms according to the rule in common Arithmetic, write them orderly under one another, with the signs of proportion; then add the logarithms of the second and third terms together, and from their sum subtract the logarithm of the first term, and the remainder will be the logarithm of the fourth term, or answer.

Or,—add together the arithmetical complement of the logarithm of the first term, and the logarithms of the second and third terms; the sum, rejecting 10 from the index, will be the logarithm of the fourth term, or term required.

N.B. The arithmetical complement of a logarithm is what it wants of 10,000000, or 20,000000, and the easiest way to find it is to begin at the left-hand, and subtract every figure from 9, except the last, which should be taken from 10; but if the index exceed 9, it must be taken from 19.—It is frequently used in the rule of Proportion and Trigonometrical calculations, to change subtractions into additions.*

* When the index is negative add it to 9, and subtract as before. And for every arithmetical complement that is added, subtract 10 from the last sum of the indices.

EXAMPLES.

1st. If a clock gain 14 seconds in 5 days 18 hours, how much will it gain in 17 days 15 hours?

5.75 days : Log. = 0.759668

17.625 days :: Log. = 1.246129
14 seconds : Log. = 1.146128

2.392257

Answer = 42″. 91 = 1.632589

Or thus; 5.75 days : Arith. Co. Log. = 9.240332
17.625 :: Log. = 1.246129
14 seconds : Log. = 1.146128

Answer = 42″.91 = 1.632589

2d. Find a fourth proportional to 98.45, 1.969, and 347.2.

98.45 : Log. = 1.993216

347.2 :: Log. = 2.540580
1.969 : Log. = 0.294246

2.834826

Answer = 6.944 = 0.841610

3d. What number will have the same proportion to .8538 as .3275 has to .0131?

.0131 : Log. = —2.117271

.3275 :: Log. = —1.515211
.8538 : Log. = —1.931356

—1.446567

Answer = 21.35 = 1.329296

4th. Required a third proportional number to 9.642 and 4.821.

9.642 : Log. = 0.984167

4.821 :: Log. = 0.683137
4.821 : Log. = 0.683137

1.366274

Answer = 2.411 = 0.382107

5. Find a fourth proportional to .05764, .7186, and .34721, by logarithms. Ans. 4.328681.

6. Find a fourth proportional to 12.687, 14.065, and 100.979, by logarithms. Ans. 112.0263.

7. Find a mean proportional between 8.76 and 43.5, by logarithms. Ans. 16.7051.

8. Find a third proportional to 12.796 and 3.24718, by logarithms. Ans. .8240216.

9. If the interest of £100 for a year, or 365 days, be £4.5, what will be the interest of £279.25 for 274 days?

Ans. £9.433294.

INVOLUTION.

To find any proposed power of a given number by Logarithms.

RULE.

Multiply the logarithm of the given number by the index of the proposed power, and the product will be the logarithm whose natural number is the power required.

When a negative index is thus multiplied, its product is negative, but what was carried from the decimal part of the logarithm must be affirmative; consequently the difference is the index of the product, which difference must be considered of the same kind with the greater, or that which was made the minuend.

EXAMPLES.

1. What is the second power of 3.874?

Log. of 3.874 = 0.588160
Index = 2

Power required = 15.01 = 1.176320

2. Required the third power of the number 2.768.

Log. of 2.768 = 0.442166
Index = 3

Answer = 21.21 = 1.326498

3. Required the second power of the number .2857.

Log. of .2857 = —1.455910
Index = 2

Answer = .08162 = — 2.911820

4. Required the third power of the number .7916.

Log. of .7916 = —1.898506
Index = 3

Answer = .4961 = —1.695518

Hence, 3 times the negative index being —3, and 2 to carry from the decimals, the difference is —1, the index of the product.

5. To find the 4th power of .09163. Ans. .000070494.
6. To find the 2d power of 6.05987. Ans. 36.72203.
7. To find the cube of 3.07146. Ans. 28.97575.
8. To find the 7th power of 1.09684. Ans. 1.909864.
9. To find the 365th power of 1.0045. Ans. 5.148888.

EVOLUTION.

To extract any proposed Root of a given number by Logarithms.

RULE.

Find the logarithm of the given number, and divide it by the index of the proposed root; the quotient is a logarithm whose natural number is the root required.

When the index of the logarithm to be divided is negative, and does not exactly contain the divisor without some remainder, increase the index by such a number as will make it exactly divisible by the index, carrying the units borrowed as so many tens to the left-hand place of the decimal, and then divide as in whole numbers.

EXAMPLES.

1. Required the square root of 847.
Index 2)2.927883 = log. of 847.

1.463941 = quot. = log. of 29.103+ = Ans.

2. Required the cube root of 847.
Index 3)2.927883 = log. of the given number.

0.975961 = quot. = log. of 9.462 = Ans. nearly.

3. Required the square root of .093.
Index 2)—2.968483 = log. of .093.

—1.484241 = quot. = log. of .304959 = Ans.

4. Required the cube root of 12345.
Index 3)4.091491 = log. of 12345.

1.363830 = quot. = log of 23.116 = Ans.

5. To find the cube root of .00048.
Power, or index 3)$\bar{4}$.6812412 = log. of the number.

Root .07829735.....$\bar{2}$.8937471 = log. of the root.

Here the divisor 3 not being exactly contained in $\bar{4}$, augment it by 2, to make it become $\bar{6}$, in which the divisor is contained just $\bar{2}$ times; and the 2 borrowed being prefixed to the other figures, makes 2.6812412, which divided by 3 gives .8937471; therefore, $\bar{2}$.8937471 is the log. of the root.

6. To find the fourth root of .967845, by logarithms. Ans. .9918624.

7. To find the cube root of 2.987635. Ans. 1.440265.

8. To find the cube root of $\frac{1}{3.141592}$. Ans. .6827842.

9. To find the value of $(.001234)^{\frac{2}{3}}$. Ans. .0115047.

10. To find the tenth root of 2. Ans. 1.071773.

SECTION IV.

ELEMENTS OF PLANE GEOMETRY.

DEFINITIONS.

See PLATE I.

1. GEOMETRY is that science wherein we consider the properties of magnitude.

2. A point is that which has no parts, being of itself indivisible; as *A*.

3. A line has length but no breadth; as *AB*, figures 1 and 2.

4. The extremities of a line are points, as the extremities of the line *AB* are the points *A* and *B*, figures 1 and 2.

5. A right line is the shortest that can be drawn between any two points, as the line *AB*, fig. 1; but if it be not the shortest, it is then called a curve line, as *AB*, fig. 2.

6. A superficies or surface is considered only as having length and breadth, without thickness, as *ABCD*, fig. 3.

7. The extremities of a superficies are lines.

8. The inclination of two lines meeting one another (provided they do not make one continued line), or the opening between them, is called an angle. Thus in fig. 4 the inclination of the

line *AB* to the line *BC*, meeting each other in the point *B*, or the opening of the two lines *BA* and *BC*, is called an angle, as *ABC*.

Note.—When an angle is expressed by three letters, the middle one is that at the angular point.

9. When the lines that form the angle are right ones, it is then called a right-lined angle, as *ABC*, fig. 4. If one of them be right and the other curved, it is called a mixed angle, as *B*, fig. 5. If both of them be curved, it is called a curved-lined or spherical angle, as *C*, fig. 6.

10. If a right line *CD* (fig. 7) fall upon another right line *AB*, so as to incline to neither side, but make the angles *ADC*, *CDB*, on each side equal to each other, then those angles are called right angles, and the line *CD* a perpendicular.

11. An obtuse angle is that which is wider or greater than a right one, as the angle *ADE*, fig. 7, and an acute angle is less than a right one, as *EDB*, fig. 7.

12. Acute and obtuse angles in general are called oblique angles.

13. If a right line *CB*, fig. 8, be fastened at the end *C*, and the other end *B* be carried quite round, then the space comprehended is called a circle; and the curve line described by the point *B* is called the circumference or the periphery of the circle; the fixed point *C* is called its centre.

14. The describing line *CB*, fig. 8, is called the semidiameter or radius; so is any line from the centre to the circumference; whence all radii of the same or of equal circles are equal.

15. The diameter of a circle is a right line drawn through the centre, and terminating in opposite points of the circumference; and it divides the circle and circumference into two equal parts, called semicircles; and is double the radius, as *AB* or *DE*, fig. 8.

16. The circumference of every circle is supposed to be divided into 360 equal parts called degrees, and each degree into 60 equal parts called minutes, and each minute into 60 equal parts called seconds, and these into thirds, fourths, &c. these parts being greater or less as the radius is.

17. A chord is a right line drawn from one end of an arc or arch (that is, any part of the circumference of a circle) to the other, and is the measure of the arc. Thus the right line *HG* is the measure of the arc *HBG*, fig 8.

18. The segment of a circle is any part thereof which is cut off by a chord: thus the space which is comprehended between the chord *HG* and the arc *HBG*, or that which is compre-

hended between the said chord *HG* and the arc *HDAEG* are called segments. Whence it is plain, fig. 8,

1. That any chord will divide the circle into two segments.
2. The less the chord is, the more unequal are the segments.
3. When the chord is greatest it becomes a diameter, and then the segments are equal; and each segment is a semicircle.*

19. A sector of a circle is a part thereof less than a semicircle, which is contained between two radii and an arc: thus the space contained between the two radii *CH*, *CB*, and the arc *HB* is a sector, fig. 8.

20. The right sine of an arc is a perpendicular line let fall from one end thereof, to a diameter drawn to the other end: thus *HL* is the right sine of the arc *HB*.

The sines on the same diameter increase till they come to the centre, and so become the radius; hence it is plain that the radius *CD* is the greatest possible sine, and thence is called the whole sine.

Since the whole sine *CD* (fig. 8) must be perpendicular to the diameter (by def. 20), therefore producing *DC* to *E*, the two diameters *AB* and *DE* cross one another at right angles, and thus the periphery is divided into four equal parts, as *BD*, *DA*, *AE*, and *EB* (by def. 10); and so *BD* becomes a quadrant, or the fourth part of the periphery; therefore the radius *DC* is always the sine of a quadrant, or of the fourth part of the circle *BD*.

Sines are said to be of as many degrees as the arc contains parts of 360: so the radius being the sine of a quadrant becomes the sine of 90 degrees, or the fourth part of the circle, which is 360 degrees.

21. The versed sine of an arc is that part of the diameter that lies between the right sine and the circumference: thus *LB* is the versed sine of the arc *HB*, fig. 8.

22. The tangent of an arc is a right line touching the periphery, being perpendicular to the end of the diameter, and is terminated by a line drawn from the centre through the other end: thus *BK* is the tangent of the arc *HB*, fig. 8.

23. And the line which terminates the tangent, that is, *CK*, is called the secant of the arc *HB*, fig. 8.

24. What an arc wants of a quadrant is called the complement thereof: thus *DH* is the complement of the arc *HB*, fig. 8.

25. And what an arc wants of a semicircle is called the sup-

* For the demonstration of this consult Prop. 15, Book III. Simpson's Euclid.

plement thereof: thus *AH* is the supplement of the arc *HB*, fig. 8.

26. The sine, tangent, or secant of the complement of any arc is called the co-sine, co-tangent, or co-secant of the arc itself: thus *FH* is the sine, *DI* the tangent, and *CI* the secant of the arc *DH*: or they are the co-sine, co-tangent, or co-secant of the arc *HB*, fig. 8.

27. The sine of the supplement of an arc is the same with the sine of the arc itself; for drawing them according to def. 30, there results the self-same line: thus *HL* is the sine of the arc *HB*, or of its supplement *ADH*, fig. 8.

28. The measure of a right-lined angle is the arc of a circle swept from the angular point, and contained between the two lines that form the angle: thus the angle *HCB*, fig. 8, is measured by the arc *HB*, and is said to contain so many degrees as the arc *HB* does; so if the arc *HB* is 60 degrees, the angle *HCB* is an angle of 60 degrees.

Hence angles are greater or less according as the arc described about the angular point, and terminated by the two sides, contains a greater or less number of degrees of the whole circle.

29. The sine, tangent, and secant of an arc is also the sine, tangent, and secant of an angle whose measure the arc is; thus, because the arc *HB* is the measure of the angle *HCB*, and since *HL* is the sine, *BK* the tangent, and *CK* the secant, *BL* the versed sine, *HF* the co-sine, *DI* the co-tangent, and *CI* the co-secant, &c. of the arc *BH*; then *HL* is called the sine, *BK* the tangent, *CK* the secant, &c. of the angle *HCB*, whose measure is the arc *HB*, fig. 8.

30. Parallel lines are such as are equidistant from each other, as *AB*, *CD*, fig. 9.

31. A figure is a space bounded by a line or lines. If the lines be right it is called a rectilineal figure; if curved it is called a curvilineal figure; but if they be partly right and partly curved lines it is called a mixed figure.

32. The most simple rectilineal figure is a triangle, being composed of three right lines, and is considered in a double capacity: 1st, with respect to its sides; and 2d, to its angles.

33. In respect to its sides, it is either equilateral, having the three sides equal, as *A*, fig. 10.

34. Or isosceles, having two equal sides, as *B*, fig. 11.

35. Or scalene, having the three sides unequal, as *C*, fig. 12.

36. In respect to its angles, it is either right-angled, having one right angle, as *D*, fig. 13,

37. Or obtuse-angled, having one obtuse angle, as *E*, fig. 14.

38. Or acute-angled, having all the angles acute, as *F*, fig. 15.

39. Acute and obtuse-angled triangles are in general called oblique-angled triangles, in all which any side may be called the base, and the other two the sides.

40. The perpendicular height of a triangle is a line drawn from the vertex to the base perpendicularly: thus if the triangle *ABC* be proposed, and *BC* be made its base, then if from the vertex *A* the perpendicular *AD* be drawn to *BC*, the line *AD* will be the height of the triangle *ABC*, standing on *BC* as its base, fig. 16.

Hence all triangles between the same parallels have the same height, since all the perpendiculars are equal from the nature of parallels.

41. Any figure of four sides is called a quadrilateral figure.

42. Quadrilateral figures, whose opposite sides are parallel, are called parallelograms: thus *ABCD* is a parallelogram, fig. 3, 17, and *AB*, fig. 18, 19.

43. A parallelogram whose sides are all equal and angles right is called a square, as *ABCD*, fig. 17.

44. A parallelogram whose opposite sides are equal and angles right is called a rectangle, or an oblong, as *ABCD*, fig. 3.

45. A rhombus is a parallelogram of equal sides, and has its angles oblique, as A, fig. 18, and is an inclined square.

46. A rhomboides is a parallelogram whose opposite sides are equal and angles oblique; as B, fig. 19, and may be conceived as an inclined rectangle.

47. Any quadrilateral figure that is not a parallelogram is called a trapezium. Plate 7, fig. 3.

48. Figures which consist of more than four sides are called polygons; if the sides are all equal to each other, they are called regular polygons. They sometimes are named from the number of their sides, as a five-sided figure is called a pentagon, one of six sides a hexagon, &c.; but if their sides are not equal to each other, then they are called irregular polygons, as an irregular pentagon, hexagon, &c.

49. Four quantities are said to be in proportion when the product of the extremes is equal to that of the means: thus if *A* multiplied by *D* be equal to *B* multiplied by *C*, then *A* is said to be to *B* as *C* is to *D*.

POSTULATES, OR PETITIONS.

1. That a right line may be drawn from any one given point to another.

2. That a right line may be produced or continued at pleasure.

3. That from any centre and with any radius the circumference of a circle may be described.

4. It is also required that the equality of lines and angles to others given, be granted as possible: that it is possible for one right line to be perpendicular to another at a given point or distance; and that every magnitude has its half, third, fourth, &c. part.

Note.—Though these postulates are not always quoted, the reader will easily perceive where and in what sense they are to be understood.

AXIOMS, OR SELF-EVIDENT TRUTHS.

1. Things that are equal to one and the same thing are equal to each other.

2. Every whole is greater than its part.

3. Every whole is equal to all its parts taken together.

4. If to equal things equal things be added, the whole will be equal.

5. If from equal things equal things be deducted, the remainders will be equal.

6. If to or from unequal things equal things be added or taken, the sums or remainders will be unequal.

7. All right angles are equal to one another.

8. If two right lines not parallel be produced towards their nearest distance, they will intersect each other.

9. Things which mutually agree with each other are equal.

NOTES.

A theorem is a proposition wherein something is proposed to be demonstrated.

A problem is a proposition wherein something is to be done or effected.

A lemma is some demonstration previous and necessary, to render what follows the more easy.

A corollary is a consequent truth, deduced from a foregoing demonstration.

A scholium is a remark or observation made upon something going before.

GEOMETRICAL THEOREMS.

THEOREM I.

PL. 1. *fig.* 20.

If a right line falls on another, as AB, or EB, does on CD, it either makes with it two right angles, or two angles equal to two right angles.

1. If AB be perpendicular to CD, then (by def. 10) the angles CBA and ABD will be each a right angle.

2. But if the line fall slantwise, as EB, and let AB be perpendicular to CD; then the $\angle DBA = DBE + EBA$: add ABC to each; then, $DBA + ABC = DBE + EBA + ABC$; but $CBE = EBA + ABC$, therefore the angles $DBE + EBC = DBA + ABC$, or two right angles. Q. E. D.

Corollary 1. Whence if any number of right lines were drawn from one point, on the same side of a right line, all the angles made by these lines will be equal to two right angles.

2. And all the angles which can be made about a point will be equal to four right angles.

THEOREM II.

PL. 1. *fig.* 21.

If one right line cross another (as AC does BD), the opposite angles made by those lines will be equal to each other: that is, AEB to CED, and BEC to AED.

By theorem 1, $BEC + CED =$ two right angles.
and $CED + DEA =$ two right angles.

Therefore (by axiom 1) $BEC + CED = CED + DEA$; take CED from both, and there remains $BEC = DEA$ (by axiom 5). Q. E. D.

After the same manner $CED + AED =$ two right angles; and $AED + AEB =$ two right angles; wherefore taking AED from both, there remains $CED = AEB$. Q. E. D.

THEOREM III.

PL. 1. *fig.* 22.

If a right line cross two parallels, as GH does AB and CD, then,

1. *Their external angles are equal to each other, that is, GEB = CFH.*

2. *The alternate angles will be equal, that is, AEF = EFD and BEF = CFE.*

3. *The external angle will be equal to the internal and opposite one on the same side, that is, GEB = EFD and AEG = CFE.*

4. *And the sum of the internal angles on the same side are equal to two right angles; that is, BEF + DFE are equal to two right angles, and AEF + CFE are equal to two right angles.*

1. Since AB is parallel to CD, they may be considered as one broad line, crossed by another line, as GH; then (by the last theo.) $GEB=CFH$, and $AEG=HFD$.

2. Also $GEB=AEF$, and $CFH=EFD$; but $GEB=CFH$ (by part 1. of this theo.), therefore $AEF=EFD$. The same way we prove $FEB=EFC$.

3. $AEF=EFD$ (by the last part of this theo.); but $AEF=GEB$ (by theo. 2), therefore $GEB=EFD$. The same way we prove $AEG=CFE$.

4. For since $GEB=EFD$, to both add FEB; then (by axiom 4) $GEB+FEB=EFD+FEB$; but $GEB+FEB$ are equal to two right angles (by theo. 1), therefore $EFD+FEB$ are equal to two right angles: after the same manner we prove that $AEF+CFE$ are equal to two right angles. Q. E. D.*

THEOREM IV.

Pl. 1. *fig.* 23.

In any triangle ABC, one of its legs, as BC, being produced towards D, it will make the external angle ACD equal to the two internal opposite angles taken together; viz. to B and A.

Through C, let CE be drawn parallel to AB; then since BD cuts the two parallel lines BA, CE, the angle $ECD=B$ (by part 3 of the last theo.); and again, since AC cuts the same parallels, the angle $ACE=A$ (by part 2 of the last), therefore $ECD+ACE=ACD=B+A$. Q. E. D.

Cor. 1. Hence, if a triangle have its exterior angle and one of its opposite interior angles double of those in another triangle, its remaining opposite interior angle will also be double of the corresponding angle in the other.†

That invaluable instrument, Hadley's Quadrant, is founded on this corollary, annexed as an obvious consequence of the theorem. A ray of light SA (*Pl.* 14. *fig.* 2) from the sun, against the mirror at A, is reflected at an angle equal to its incidence; and now striking the half-silvered glass at C, it is again reflected to E, where the eye likewise receives, through the transparent part of that glass, a direct ray from the boundary of the horizon.

Hence, the triangle AEC has its exterior angle ECD and one of its interior angles CAE respectively double of the exterior angle BCD and the interior angle CAB of the triangle

* For an excellent demonstration of this theorem (by the motion of the straight line crossing the parallel lines about a point in one of them), the reader will consult Leslie's Geometry, Prop. 23, page 26.

† This corollary, with the following demonstration, is found in Leslie's Geometry, pages 32 and 406.

ABC; wherefore the remaining interior angle AEC, or SEZ, is double of ABC; that is, the altitude of the sun above the horizon is double of the inclination of the two mirrors. But the glass at C remaining fixed, the mirror at A is attached to a moveable index, which marks their inclination.

The same instrument, in its most improved state, and fitted with a telescope, forms the sextant, which, being admirably calculated for measuring angles in general, has rendered the most important services to geography and navigation.

THEOREM V.

PL. 1. *fig.* 23.

In any triangle ABC, all the three angles, taken together, are equal to two right angles, viz. $A+B+ACB=$ *two right angles.*

Produce CB to any distance, as D, then (by the last) $ACD=B+A$; to both add ACB; then $ACD+ACB=A+B+ACB$; but $ACD+ACB=$ two right angles (by theo. 1); therefore the three angles $A+B+ACB=$ two right angles. Q. E. D.

Cor. 1. Hence if one angle of a triangle be known, the sum of the other two is also known; for since the three angles of every triangle contain two right ones, or 180 degrees, therefore 180— the given angle will be equal to the sum of the other two; or 180— the sum of two given angles gives the other one.

Cor. 2. In every right-angled triangle, the two acute angles are = 90 degrees, or to one right angle; therefore 90— one acute angle gives the other.

THEOREM VI.

PL. 1. *fig.* 24.

If in any two triangles, ABC, DEF, there be two sides AB, AC in the one severally equal to DE, DF in the other, and the angle A contained between the two sides in the one equal to D in the other; then the remaining angles of the one will be severally equal to those of the other, viz. $B=E$, *and* $C=F$; *and the base of the one BC will be equal to EF, that of the other.*

If the triangle ABC be supposed to be laid on the triangle DEF, so as to make the points A and B coincide with D and E, which they will do, because $AB=DE$ (by the hypothesis); and since the angle $A=D$, the line AC will fall along DF, and inasmuch as they are supposed equal, C will fall in F; seeing therefore the three points of one coincide with those of the other triangle, they are manifestly equal to each other; therefore the angle $B=E$, and $C=F$, and $BC=EF$. Q. E. D.

LEMMA.

PL. 1. *fig.* 11.

If two sides of a triangle abc be equal to each other, that is, ac=cb, the angles which are opposite to those equal sides will also be equal to each other; viz. a=b.

For let the triangle *abc* be divided into two triangles *acd*, *dcb*, by making the angle *acd=dcb* (by postulate 4); then because *ac=bc*, and *cd* common (by the last), the triangle *adc=dcb*; and therefore the angle *a=b*. Q. E. D.

Cor. Hence if from any point in a perpendicular which bisects a given line there be drawn right lines to the extremities of the given one, they with it will form an isosceles triangle.

THEOREM VII.

PL. 1. *fig.* 25.

The angle BCD at the centre of a circle ABED is double the angle BAD at the circumference, standing upon the same arc BED.

Through the point *A*, and the centre *C*, draw the line *ACE*; then the angle *ECD=CAD+CDA* (by theo. 4); but since *AC=CD*, being radii of the same circle, it is plain (by the preceding lemma) that the angles subtended by them will be also equal, and that their sum is double to either of them, that is, *DAC+ADC* is double to *CAD*, and therefore *ECD* is double to *CAD*; after the same manner *BCE* is double to *CAB*, wherefore *BCE+ECD*, or *BCD*, is double to *BAC +CAD*, or to *BAD*. Q. E. D.

Cor. 1. Hence an angle at the circumference is measured by half the arc it subtends or stands on.

Fig. 26.

Cor. 2. Hence all angles at the circumference of a circle which stand on the same chord as *AB* are equal to each other, for they are all measured by half the arc they stand on, viz. by half the arc *AB*

Fig. 26.

Cor. 3. Hence an angle in a segment greater than a semicircle is less than a right angle; thus *ADB* is measured by half the arc *AB*; but as the arc *AB* is less than a semicircle, therefore half the arc *AB*, or the angle *ADB*, is less than half a semicircle, and consequently less than a right angle.

Fig. 27.

Cor. 4. An angle in a segment less than a semicircle is greater than a right angle; for since the arc *AEC* is greater than a semicircle, its half, which is the measure of the angle *ABC*,

must be greater than half a semicircle, that is, greater than a right angle.

Fig. 28.

Cor. 5. An angle in a semicircle is a right angle, for the measure of the angle ABD is half of a semicircle AED, and therefore a right angle.

THEOREM VIII.

PL. 1. *fig.* 29.

If from the centre C of a circle ABE there be let fall the perpendicular CD on the chord AB, it will bisect it in the point D.

Let the lines AC and CB be drawn from the centre to the extremities of the chord; then since $CA=CB$, the angles $CAB=CBA$ (by the lemma). But the triangles ADC, BDC are right-angled ones, since the line CD is a perpendicular; and so the angle $ACD=DCB$ (by cor. 2, theo. 5); then have we AC, CD, and the angle ACD in one triangle severally equal to CB, CD, and the angle BCD in the other; therefore (by theo. 6) $AD=DB$. Q. E. D.

Cor. Hence it follows, that any line bisecting a chord at right angles is a diameter; for a line drawn from the centre perpendicular to a chord bisects that chord at right angles; therefore, conversely, a line bisecting a chord at right angles must pass through the centre, and consequently be a diameter.

THEOREM IX.

PL. 1. *fig.* 29.

If from the centre of a circle ABE there be drawn a perpendicular CD on the chord AB, and produced till it meets the circle in F, that line CF will bisect the arc AB in the point F.

Let the lines AF and BF be drawn; then in the triangles ADF, BDF, $AD=BD$ (by the last); DF is common, and the angle $ADF=BDF$, being both right, for CD or DF is a perpendicular. Therefore (by theo. 6) $AF=FB$; but in the same circle, equal lines are chords of equal arcs, since they measure them (by def. 19); whence the arc $AF=FB$, and so AFB is bisected in F by the line CF.

Cor. Hence the sine of an arc is half the chord of twice that arc. For AD is the sine of the arc AF (by def. 20), AF is half the arc, and AD half the chord AB (by theo. 8); therefore the corollary is plain.

THEOREM X.

PL. 1. *fig.* 30.

In any triangle ABD, the half of each side is the sine of the opposite angle.

Let the circle *ADB* be drawn through the points *A*, *B*, *D*; then the angle *DAB* is measured by half the arc *BKD* (by cor. 1, theo. 7), *viz.* the arc *BK* is the measure of the angle *BAD*; therefore (by cor. to the last) *BE*, the half of *BD*, is the sine of *BAD*: in the same way may be proved that half of *AD* is the sine of *ABD*, and the half of *AB* the sine of *ADB*. Q. E. D.

THEOREM XI.

PL. 1. *fig.* 22.

If a right line GH cut two other right lines AB, CD, so as to make the alternate angles AEF, EFD equal to each other, then the lines AB and CD will be parallel.

If it be denied that *AB* is parallel to *CD*, let *IK* be parallel to it; then $IEF=(EFD)=AEF$ (by part 2, theo. 3), a greater to a less, which is absurd, whence *IK* is not parallel; and the like we can prove of all other lines but *AB*; therefore *AB* is parallel to *CD*. Q. E. D.

THEOREM XII.

PL. 1. *fig.* 3.

If two equal and parallel lines AB, CD, be joined by two other lines AD, BC, those shall be also equal and parallel.

Let the diameter or diagonal *BD* be drawn, and we will have the triangles *ABD*, *CBD*, whereof *AB* in one is = to *CD* in the other, *BD* common to both, and the angle $ABD=CDB$ (by part 2, theo. 3); therefore (by theo. 6) $AD=CB$, and the angle $CBD=ADB$; and thence the lines *AD* and *BC* are parallel, by the preceding theorem.

Cor. 1. Hence the quadrilateral figure *ABCD* is a parallelogram, and the diagonal *BD* bisects the same, inasmuch as the triangle $ABD=BCD$, as now proved.

Cor. 2. Hence also the triangle *ABD* on the same base *AB*, and between the same parallels with the parallelogram *ABCD*, is half the parallelogram.

Cor. 3. It is hence also plain that the opposite sides of a parallelogram are equal; for it has been proved that, *ABCD* being a parallelogram, *AB* will be = *CD*, and $AD=BC$.

THEOREM XIII.

PL. 1. *fig.* 31

All parallelograms on the same or equal bases and between the same parallels are equal to one another; that is, if BD=GH, and the lines BH and AF are parallel, then the parallelogram ABDC=BDFE=EFHG.

For $AC=BD=EF$ (by cor. the last); to both add CE, then $AE=CF$. In the triangles ABE, CDF, $AB=CD$ and $AE=CF$, and the angle $BAE=DCF$ (by part 3, theo. 3); therefore the triangle $ABE=CDF$ (by theo. 6); let the triangle CKE be taken from both, and we will have the trapezium $ABKC=KDFE$; to each of these add the triangle BKD, then the parallelogram $ABCD=BDEF$: in like manner we may prove the parallelogram $EFGH=BDEF$. Wherefore $ABDC=BDEF=EFGH$. Q. E. D.

Cor. Hence it is plain that triangles on the same or equal bases and between the same parallels are equal, seeing (by cor. 2, theo. 12) they are the halves of their respective parallelogram.

THEOREM XIV.

PL. 1. *fig.* 32.

In every right-angled triangle, ABC, the square of the hypothenuse or longest side, BC, or BCMH, is equal to the sum of the squares made on the other two sides AB and AC, that is, ABDE and ACGF.

Through A draw AKL perpendicular to the hypothenuse BC, join AH, AM, DC, and BG; in the triangles BDC, ABH, $BD=BA$, being sides of the same square, and also $BC=BH$, and the included angles $DBC=ABH$ (for $DBA=CBH$ being both right, to both add ABC, then $DBC=ABH$), therefore the triangle $DBC=ABH$ (by theo. 6); but the triangle DBC is half of the square $ABDE$ (by cor. 2, theo. 12), and the triangle ABH is half the parallelogram $BKLH$. The same way it may be proved that the square $ACGF$ is equal to the parallelogram $KCLM$. So $ABDE+ACGF$ the sum of the squares $=BKLH+KCML$, the sum of the two parallelograms or square $BCMH$; therefore the sum of the squares on AB and AC is equal to the square on BC. Q. E. D.*

Cor. 1. Hence the hypothenuse of a right-angled triangle may be found by having the sides: thus, the square root of the sum of the squares of the base and perpendicular will be the hypothenuse.

* For different demonstrations of this excellent theorem, the reader may consult Leslie's Geometry (Prop. xi. book ii.).

Cor. 2. Having the hypothenuse and one side given to find the other; the square root of the difference of the squares of the hypothenuse and given side will be the required side.

THEOREM XV.

Pl. 1. *fig.* 33.

In all circles the chord of 60 *degrees is always equal in length to the radius.*

Thus in the circle AEBD, if the arc AEB be an arc of 60 *degrees, and the chord AB be drawn, then AB=CB=AC.*

In the triangle ABC the angle ACB is 60 degrees, being measured by the arc AEB; therefore the sum of the other two angles is 120 degrees (by cor. 1, theo. 5); but since $AC=CB$, the angle $CAB=CBA$ (by lemma preceding theo. 7); consequently each of them will be 60, the half of 120 degrees, and the three angles will be equal to one another as well as the three sides: wherefore $AB=BC=AC$. Q. E. D.

Cor. Hence the radius from whence the lines on any scale are formed is the chord of 60 degrees on the line of chords.

THEOREM XVI.

Pl. 1. *fig.* 34.

If in two triangles, ABC, abc, all the angles of one be each respectively equal to all the angles of the other; that is, A=a, B=b, C=c; then the sides opposite to the equal angles will be proportional, viz.

$$AB : ab :: AC : ac$$
$$AB : ab :: BC : bc$$
$$\text{and } AC : ac :: BC : bc$$

For the triangles being inscribed in two circles, it is plain, since the angle $A=a$, the arc $BDC=$* bdc and consequently the chord BC is to bc as the radius of the circle ABC is to the radius of the circle abc (for the greater the radius is, the greater is the circle described by that radius; and consequently the greater any particular arc of that circle is, so the chord, sine, tangent, &c. of that arc will be also greater. Therefore, in general, the chord, sine, tangent, &c. of any arc is proportional to the radius of the circle); the same way the chord AB is to the chord ab in the same proportion. So $AB : ab :: BC : bc$. The same way the rest may be proved to be proportional.

* The arc BDC is not = in length to bdc, as might be supposed from the sign of equality; but they contain the same number of degrees, as being the measure of equal angles.

THEOREM XVII.

PL. 1. *fig.* 35.

If from a point A without a circle DBCE there be drawn two lines ADE, ABC, each of them cutting the circle in two points, the product of one whole line into its external part, viz. AC into AB, will be equal to that of the other line into its external part, viz. AE into AD.

Let the lines DC, BE be drawn into two triangles ABE, ADC; the angle $AEB=ACD$ (by cor. 2, theo. 7); the angle A is common, and (by cor. 1, theo. 5) the angle $ADC=ABE$; therefore the triangles ABE, ADC are mutually equiangular, and consequently (by the last) $AC : AE :: AD : AB$; wherefore AC multiplied by AB will be equal to AE multiplied by AD. Q. E. D.

THEOREM XVIII.

PL. 2. *fig.* 1.

Triangles ABC, BCD, and parallelograms ABCF and BDEC, having the same altitude, have the same proportion between themselves as their bases BA and BD.

Let any aliquot part of AB be taken which will also measure BD: suppose that to be Ag, which will be contained twice in AB, and three times in BD, the parts Ag, gB, Bh, hi, and iD being all equal, and let the lines gC, hC, and iC be drawn: then (by cor. to theo. 13) all the small triangles AgC, gCB, BCh, &c. will be equal to each other, and will be as many as the parts into which their bases were divided; therefore it will be, as the sum of the parts in one base is to the sum of those in the other, so will be the sum of the small triangles in the first to the sum of the small triangles in the second triangle; that is, $AB : BD :: ABC : BDC$.

Whence also the parallelograms $ABCF$ and $BDEC$, being (by cor. 2, theo. 12) the doubles of the triangles, are likewise as their bases. Q. E. D.

Note.—Wherever there are several quantities connected with the sign (: :) the conclusion is always drawn from the first two and last two proportionals.

THEOREM XIX.

PL. 2. *fig.* 2.

Triangles ABC, DEF, standing upon equal bases AB and DE, are to each other as their altitudes CG and FH.

Let BI be perpendicular to AB and equal to CG, in which let $KB=FH$, and let AI and AK be drawn.

The triangle $AIB=ACB$ (by cor. to theo. 13), and $AKB=DEF$; but (by theo. 18) $BI : BK :: ABI : ABK$. That is, $CG : FH :: ABC : DEF$. Q. E. D.

THEOREM XX.

PL. 2. *fig.* 3.

If a right line BE be drawn parallel to one side of a triangle ACD, it will cut the two other sides proportionally, viz. AB : BC :: AE : ED.

Draw CE and BD; the triangles BEC and EBD being on the same base BE and under the same parallel CD, will be equal (by cor. to theo. 13) therefore (by theo. 18) $AB : BC :: (BEA : BEC$ or $BEA : BED ::) AE : ED$. Q. E. D.

Cor. 1. Hence also $AC : AB :: AD : AE$; for $AC : AB (AEC : AEB :: ABD : AEB) :: AD : AE$.

Cor. 2. It also appears that a right line which divides two sides of a triangle proportionally must be parallel to the remaining side.

Cor. 3. Hence, also, theo. 16 is manifest; since the sides of the triangles ABE, ACD, being equiangular, are proportional.

THEOREM XXI.

PL. 2. *fig.* 4.

If two triangles ABC, ADE have an angle BAC in the one equal to an angle DAE in the other, and the sides about the equal angles proportional; that is, AB : AD :: AC : AE; then the triangles will be mutually equiangular.

In AB take $Ad=AD$, and let de be parallel to BC, meeting AC in e.

Because (by the first cor. to the foregoing theo.) $AB : Ad$ (or AD) $:: AC : Ae$, and (by the hypothesis, or what is given in the theorem) $AB : AD :: AC : AE$; therefore $Ae=AE$, seeing AC bears the same proportion to each; and (by theo. 6) the triangle $Ade=ADE$, therefore the angle $Ade=D$ and $Aed=E$; but since ed and BC are parallel (by part 3, theo. 3) $Ade=B$, and $Aed=C$, therefore $B=D$ and $C=E$. Q. E. D.

THEOREM XXII.

PL. 2. *fig.* 5.

Equiangular triangles ABC, DEF are to one another in a duplicate proportion of their homologous or like sides; or as the squares AK and DM of their homologous sides.

Let the perpendiculars CG and FH be drawn, as well as the diagonals BI and EL.

The perpendiculars make the triangles *ACG* and *DFH* equiangular, and therefore similar (by theo. 16); for because the angle $CAG=FDH$, and the right angle $AGC=DHF$, the remaining angle $ACG=DFH$ (by cor. 2. theo. 5).

Therefore $GC : FH :: (AC : DF ::) AB : DE$, or, which is the same thing, $GC : AB :: FH : DE$, for *FH* multiplied by $AB=GC$ multiplied by *DE*.

By theo. 19, $ABC : ABI :: (CG$: *AI* or *AB* as before :: $FH : DE$, or *DL*): : $DFE : DLE$, therefore $ABC : ABI :: DFE : DLE$, or $ABC : AK :; DFE : DM$, for *AK* is double the triangle *ABI*, and *DM* double the triangle *DEL*, (by cor. 2. theo. 12.) Q. E. D.

THEOREM XXIII.

Pl. 2. *fig.* 6.

Like polygons ABCDE, abcde are in a duplicate proportion to that of the sides AB, ab, which are between equal angles A and B and a and b, or as the squares of the sides AB, ab.

Draw *AD*, *AC*; *ad*, *ac*.

By the hypothesis $AB : ab :: BC : bc$, and thereby also the angle $B=b$; therefore (by theo. 21) $BAC=bac$; and $ACB=acb$: in like manner $EAD=ead$, and $EDA=eda$. If therefore from the equal angles *A* and *a*, we take the equal ones $EAD+BAC=ead+bac$, the remaining angle $DAC=dac$, and if from the equal angles *D* and *d*, $EDA=eda$ be taken, we shall have $ADC=adc$: and in like manner if from *C* and *c* be taken $BCA=bca$, we shall have $ACD=acd$; and so the respective angles in every triangle will be equal to those in the other.

By theo. 22, $ABC : abc$: : the square of *AC* to the square of *ac*, and also $ADC : adc$: : the square of *AC* to the square of *ac*; therefore, from equality of proportions, $ABC : abc :: ADC : adc$; in like manner we may show that $ADC : adc :: EAD : ead$. Therefore it will be, as one antecedent is to one consequent, so are all the antecedents to all the consequents. That is, *ABC* is to *abc* as the sum of the three triangles in the first polygon is to the sum of those in the last. Or *ABC* will be to *abc* as polygon to polygon.

The proportion of *ABC* to *abc* (by the foregoing theo.) is as the square of *AB* is to the square of *ab*, but the proportion of polygon to polygon is as *ABC* to *abc*, as now shown: therefore the proportion of polygon to polygon is as the square of *AB* to the square of *ab*.

THEOREM XXIV.

Pl. 1. *fig.* 8.

Let DHB be a quadrant of a circle described by the radius CB; HB an arc of it, and DH its complement; HL or FC the sine, FH or CL its co-sine, BK its tangent, DI its cotangent; CK its secant, and CI its co-secant. Fig. 8.

1. The cosine of an arc is to the sine as the radius is to the tangent.

2. The radius is to the tangent of an arc as the cosine of it is to the sine.

3. The sine of an arc is to its cosine as the radius to its cotangent.

4. Or the radius is to the cotangent of an arc as its sine to its co-sine.

5. The cotangent of an arc is to the radius as the radius to the tangent.

6. The cosine of an arc is to the radius as the radius is to the secant.

7. The sine of an arc is to the radius as the tangent is to the secant.

The triangles *CLH* and *CBK* being similar (by theo. 16),

1. *CL* : *LH* : : *CB* : *BK*.

2. Or, *CB* : *BK* : : *CL* : *LH*.

The triangles *CFH* and *CDI* being similar,

3. *CF* (or *LH*) : *FH* : : *CD* : *DI*.

4. *CD* : *DI* : : *CF* (or *LH*) : *FH*.

The triangles *CDI* and *CBK* are similar; for the angle *CID*=*KCB*, being alternate ones (by part 2, theo. 3), the lines *CB* and *DI* being parallel, the angle *CDI*=*CBK* being both right, and consequently the angle *DCI*=*CKB*, wherefore,

5. *DI* : *CD* : : *CB* : *BK*.

And again, making use of the similar triangle *CLH* and *CBK*,

6. *CL* : *CB* : : *CH* : *CK*.

7. *HL* : *CH* : : *BK* : *CK*.

GEOMETRICAL PROBLEMS.

PROBLEM I.

Pl. 2. *fig.* 7.

To make a triangle of three given right lines BO, LB, LO, of which any two must be greater than the third.

Lay *BL* from *B* to *L*; from *B* with the line *BO* describe an arc, and from *L* with *LO* describe another arc; from *O*, the intersecting point of those arcs, draw *BO* and *OL*, and *BOL* is the triangle required.

This is manifest from the construction.

PROBLEM II.

Pl. 2. *fig.* 8.

At a point B in a given right line BC, to make an angle equal to a given angle A.

Draw any right line *ED* to form a triangle, as *EAD*, take *BF* $=AD$, and upon *BF* make the triangle *BFG*, whose side $BG=AE$, and $GF=ED$ (by the last), then also the angle $B=A$; if we suppose one triangle be laid on the other, the sides will mutually agree with each other, and therefore be equal; for if we consider these two triangles to be made of the same three given lines, they are manifestly one and the same triangle.

Otherwise,

Upon the centres *A* and *B*, at any distance, let two arcs *DE*, *FG*, be described; make the arc $FG=DE$, and through *B* and *G* draw the line *BG*, and it is done.

For since the chords *ED*, *GF* are equal, the angles *A* and *B* are also equal, as before (by def. 17).

PROBLEM III.

Pl. 2. *fig.* 9.

To bisect or divide into two equal parts any given right-lined angle BAC.

In the lines *AB* and *AC*, from the point *A*, set off equal distances $AE=AD$; then, with any distance more than the half of *DE*, describe two arcs to cut each other in some point *F*; and the right line *AF*, joining the points *A* and *F*, will bisect the given angle *BAC*.

For if *DF* and *FE* be drawn, the triangles *ADF*, *AEF* are equilateral to each other, viz: $AD=AE$, $DF=FE$, and *AF* common, wherefore $DAF=EAF$, as before.

PROBLEM IV.

PL. 2. *fig.* 10.

To bisect a right line AB.

With any distance more than half the line from *A* and *B*, describe two circles *CFD*, *CGD*, cutting each other in the points *C* and *D*; draw *CD* intersecting *AB* in *E*, then *AE*=*EB*.

For, if *AC*, *AD*, *BC*, *BD* be drawn, the triangles *ACD*, *BCD* will be mutually equilateral, and consequently the angle *ACE*=*BCE*; therefore the triangle *ACE*, *BCE*, having *AC*=*BC*, *CE* common, and the angle *ACE*=*BCE*; (by theo. 6) the base *AE*=the base *BE*.

Cor. Hence it is manifest that *CD* not only bisects *AB*, but is perpendicular to it (by def. 10).

PROBLEM V.

PL. 2. *fig.* 11.

On a given point A, in a right line EF, to erect a perpendicular.

From the point *A* lay off on each side the equal distances *AC*, *AD*; and from *C* and *D* as centres, with any interval greater than *AC* or *AD*, describe two arcs intersecting each other in *B*; from *A* to *B* draw the line *AB*, and it will be the perpendicular required.

For let *CB* and *BD* be drawn, then the triangles *CAB*, *DAB* will be mutually equilateral and equiangular, so *CAB*=*DAB*, a right angle (by def. 10).

PROBLEM VI.

PL. 2. *fig.* 12.

To raise a perpendicular on the end B of a right line AB.

From any point *D* not in the line *AB*, with the distance from *D* to *B*, let a circle be described cutting *AB* in *E*; draw from *E* through *D* the right line *EDC*, cutting the periphery in *C*, and join *CB*, and that is the perpendicular required.

EBC being a semicircle, the angle *EBC* will be a right angle (by cor. 5, theo. 7).

ROBLEM VII.

PL. 2. *fig.* 13.

From a given point A, to let fall a perpendicular upon a given right line BC.

From any point *D*, in the given line, take the distance to the given point *A*, and with it describe a circle *AGE*, make *GE*=

AG, join the points *A* and *E* by the line *AFE*, and *AF* will be the perpendicular required.

Let *DA*, *DE* be drawn, the angle *ADF*=*FDE*, *DA*=*DE*, being radii of the same circle, and *DF* common; therefore (by theo. 6) the angle *DFA*=*DFE*, and *FA* a perpendicular. (By def. 10.)

PROBLEM VIII.

Pl. 2. *fig.* 14.

Through a given point A to draw a right line AB, parallel to a given right line CD.

From the point *A* to any point *F* in the line *CD* draw the line *AF*; with the interval *FA*, and one foot of the compasses in *F*, describe the arc *AE*, and with the like interval and one foot in *A* describe the arc *BF*, making *BF*=*AE*; through *A* and *B* draw the line *AB*, and it will be parallel to *CD*.

By prob. 2, The angle *BAF*=*AFE*, and by theo. 11, *BA* and *CD* are parallel.

PROBLEM IX.

Pl. 1. *fig.* 17.

Upon a given line AB to describe a square ABCD.

Make *BC* perpendicular and equal to *AB*, and from *A* and *C*, with the line *AB* or *BC*, let two arcs be described, cutting each other in *D*; from whence to *A* and *C* let the lines *AD*, *DC* be drawn; so is *ABCD* the square required.

For all the sides are equal by construction; therefore the triangles *ADC* and *BAC* are mutually equilateral and equiangular, and *ABCD* is an equilateral parallelogram, whose angles are right. For *B* being right, *D* is also right, and *DAC*, *DCA*, *BAC*, *ACB*, each half a right angle (by lemma preceding theo. 7, and cor. 2, theo. 5), whence *DAB* and *BCD* will each be a right angle, and (by def. 43) *ABCD* is a square.

SCHOLIUM.

By the same method a rectangle or oblong may be described, the sides thereof being given.

PROBLEM X.

Pl. 2. *fig.* 15.

To divide a given right line AB into any proposed number of equal parts.

Draw the indefinite right line *AP*, making any angle with

AB, also draw *BQ* parallel to *AP*, in each of which let there be taken as many equal parts *AM*, *MN*, &c. *Bo*, *on*, &c. as you would have *AB* divided into; then draw *Mm*, *Nn*, &c. intersecting *AB* in *E*, *F*, &c. and it is done.

For *MN* and *mn* being equal and parallel, *FN* will be parallel to *EM*, and in the same manner *GO* to *FN* (by theo. 12); therefore *AM*, *MN*, *NO*, being all equal by construction, it is plain (from theo. 20) that *AE*, *EF*, *FG*, &c. will likewise be equal.

PROBLEM XI.

Pl. 2. *fig.* 16.

To find a third proportional to two given right lines A and B.

Draw two indefinite blank lines *CE*, *CD* anywise to make any angle. Lay the line *A* from *C* to *F*, and the line *B* from *C* to *G*, and draw the line *FG*; lay again the line *A* from *C* to *H*, and through *H* draw *HI* parallel to *FG* (by prob. 8), so is *CI* the third proportional required.

For, by cor. 1, theo. 20, $CG : CH :: CF : CI$.

Or, $B : A :: A : CI$.

PROBLEM XII.

Pl. 2. *fig.* 17.

Three right lines A, B, C given, to find a fourth proportional.

Having made an angle *DEF* anywise, by two indefinite blank right lines *ED*, *EF*, as before; lay the line *A* from *E* to *G*, the line *B* from *E* to *I*, and draw the line *IG*; lay the line *C* from *E* to *H*, and (by prob. 8) draw *HK* parallel thereto, so will *EK* be the fourth proportional required.

For, by cor. 1, theo. 20, $EG : EI :: EH : EK$.

Or, $A : B :: C : EK$.

PROBLEM XIII.

Pl. 3. *fig.* 1.

Two right lines A and B given, to find a mean proportional.

Draw an indefinite straight line, on which place $AB = A$ and $BC = B$; bisect *AC* (by prob. 4) in *E*, and describe the semicircle *ADC*, and from the point *B* erect the perpendicular *BD* (by prob. 5), then *BD* is a mean proportional.

For if the lines *AD*, *DC* be drawn, the angle *ADC* is a right angle (by cor. 5, theo. 7), being an angle in a semicircle.

The angles *ABD*, *DBC* are right ones (by def. 10), the line *BD* being a perpendicular; wherefore the triangles *ABD*,

DBC are similar: thus the angle *ABD*=*DBC*, being both right, the angle *DAC* is the complement of *BDA* to a right angle (by cor. 2, theo. 5), and is therefore equal to *BDC*, the angle *ADC* being a right angle as before; consequently (by cor. 1, theo. 5) the angle *ADB*=*DCB*; wherefore (by theo. 16),

AB : *BD* : : *BD* : *BC*

Or, *A* : *BD* : : *BD* : *B*.

PROBLEM XIV.

Pl. 3. *fig.* 2.

To divide a right line AB in the point E, so that AE shall have the same proportion to EB as two given lines C and D have.

Draw an indefinite blank line *AF* to the extremity of the line *AB*, to make with it any angle; lay the line *C* from *A* to *C*, and *D* from *C* to *D*, and join the points *B* and *D* by the line *BD*; through *C* draw *CE* parallel to *BD* (by prob. 8), so is *E* the point of division.

For, by theo. 20, *AC* : *CD* : : *AE* : *EB*.

Or, *C* : *D* : : *AE* : : *EB*.

PROBLEM XV.

Pl. 3. *fig.* 3.

To describe a circle about a triangle ABC, or (which is the same thing) through any three points A, B, C, which are not situated in a right line.

By prob. 4. Bisect the line *AC* by the perpendicular *DE*, and also *CB* by the perpendicular *FG*, the point of intersection *H* of these perpendiculars is the centre of the circle required; from which take the distance to any of the three points *A*, *B*, *C*, and describe the circle *ABC*, and it is done.

For, by cor. to theo. 8, the lines *DE* and *FG* must each pass through the centre; therefore their point of intersection *H* must be the centre.

SCHOLIUM.

By this method the centre of a circle may be found, by having only a segment of it given.

PROBLEM XVI.

Pl. 3. *fig.* 4.

To make an angle of any number of degrees at the point A of the line AB, suppose of 45 degrees.

From a scale of chords take 60 degrees, for 60° is equal to the radius (by cor. theo. 15), and with that distance from *A* as a centre, describe a circle from the line *AB*; take 45 degrees,

the quantity of the given angle, from the same scale of chords, and lay it on that circle from a to b; through A and b draw the line AbC, and the angle A will be an angle of 45 degrees, as required.

If the given angle be more than 90°, take its half (or divide it into any two parts less than 90 and lay them after each other on the arc, which is described with the chord of 60 degrees; through the extremity of which and the centre, let a line be drawn, and that will form the angle required, with the given line.

PROBLEM XVII.

PL. 3. *fig.* 5.

To measure a given angle ABC.

If the lines which include the angle be not as long as the chord of 60° on your scale, produce them to that or a greater length, and between them so produced, with the chord of 60° from B, describe the arc ed; which distance ed, measured on the same line of chords, gives the quantity of the angle ABC, as required; this is plain from def. 17.

PROBLEM XVIII.

PL. 3. *fig.* 6.

To make a triangle BCE equal to a given quadrilateral figure ABCD.

Draw the diagonal AC, and parallel to it (by prob. 8) DE, meeting AB produced in E; then draw CE, and ECB will be the triangle required.

For the triangles ADC, AEC being upon the same base AC, and under the same parallel ED (by cor. to theo. 13), will be equal, therefore if ABC be added to each, then $ABCD = BEC$.

PROBLEM XIX.

PL. 3. *fig.* 7.

To make a triangle DFH equal to a given five-sided figure ABCDE.

Draw DA and DB, and also EH and CF parallel to them, (by prob. 8) meeting AB produced in H and F; then draw DH, DF, and the triangle HDF is the one required.

For the triangle $DEA = DHA$, and $DBC = DFB$ (by cor. to theo. 13); therefore by adding these equations, $DEA + DBC = DHA + DFB$, if to each of these ADB be added; then $DEA + ADB + DBC = ABCDE\ (= DHA + ABD + DFB) = DHF$.

PROBLEM XX.

Pl. 3. *fig.* 8.

To project the lines of chords, sines, tangents, and secants with any radius.

On the line *AB*, let a semicircle *ADB* be described; let *CDF* be drawn perpendicular to this line from the centre *C*; and the tangent *BE* perpendicular to the end of the diameter; let the quadrants *AD*, *DB* be each divided into nine equal parts, every one of which will be ten degrees; if then from the centre *C* lines be drawn through 10, 20, 30, 40, &c. the divisions of the quadrant *BD*, and continued to *BE*, we shall there have the tangents of 10, 20, 30, 40, &c. and the secants *C* 10, *C* 20, *C* 30, &c. are transferred to the line *CF*, by describing the arcs 10, 10; 20, 20; 30, 30, &c. If from 10, 20, 30, &c. the divisions of the quadrant *BD*, there be let fall perpendiculars, let these be transferred to the radius *CB*, and we shall have the sines of 10, 20 30, &c. and if from *A* we describe the arcs 10, 10; 20, 20; 30, 30, &c. from every division of the arc *AD*, we shall have a line of chords. The same way we may have the sine, tangent, &c. to every single degree on the quadrant, by subdividing each of the nine former divisions into ten equal parts. By this method the sines, tangents, &c. may be drawn to any radius; and then, after they are transferred to lines on a rule, we shall have the scales of sines, tangents, &c. ready for use.

MATHEMATICAL
DRAWING INSTRUMENTS.

The strictness of geometrical demonstration admits of no other instruments than a rule and a pair of compasses. But, in proportion as the practice of geometry was extended to the different arts, either connected with or dependent upon it, new instruments became necessary, some to answer peculiar purposes, some to facilitate operation, and others to promote accuracy.

As almost every artist whose operations are connected with mathematical designing furnishes himself with a case of drawing instruments suited to his peculiar purposes, they are fitted up in various modes, some containing more, others fewer instruments. The smallest collection put into a case consists of a plane scale, a pair of compasses with a moveable leg, and two spare points, which may be applied occasionally to the

compasses; one of these points is to hold ink; the other a portcrayon, for holding a piece of black-lead pencil.

What is called a full pocket case, contains the following in struments.

A pair of large compasses with a moveable point, an ink point, a pencil point, and one for dotting; either of those points may be inserted in the compasses instead of the moveable leg.

A pair of plain compasses somewhat smaller than those with the moveable leg.

A pair of bow compasses.

A drawing pen with a protracting pin in the upper part.

A sector.

A plain scale.

A protractor.

A parallel rule.

A pencil and screwdriver.*

* Large collections are called *magazine cases of instruments*; these generally contain—

A pair of six inch compasses with a moveable leg, an ink point, a dotting point, the crayon point, so contrived as to hold a whole pencil, two additional pieces to lengthen occasionally one leg of the compasses, and thereby enable them to measure greater extents, and describe circles of a larger radius.

A pair of hair compasses.

A pair of bow compasses.

A pair of triangular compasses.

A sector.

A parallel rule.

A protractor.

A pair of proportional compasses, either with or without an adjusting screw.

A pair of wholes and halves.

Two drawing pens, and a pointril.

A pair of small hair compasses, with a head similar to those of the bow compasses.

A knife, a file, a key, and screwdriver, or the compasses in one piece.

A small set of fine water-colours.

To these some of the following instruments are often added:—

A pair of beam compasses.

A pair of gunners' callipers.

A pair of elliptical compasses.

A pair of spiral compasses.

A pair of perspective compasses.

A pair of compasses with a micrometer screw.

A rule for drawing lines, tending to a centre at a great distance.

A protractor and parallel rule.

One or more parallel rules.

A pantographer, or pentagraph.

A pair of sectoral compasses, forming at the same time a pair of beam and calliper compasses.

In a case with the best instruments, the protractor and plain scale are always combined. The instruments in most general use are those of six inches; instruments are seldom made longer, but often smaller. Those of six inches are, however, to be preferred, in general, before any other size; they will effect all that can be performed with the shortest ones, while, at the same time, they are better adapted to large works.

OF DRAWING COMPASSES.

Compasses are made either of silver or brass, but with steel points. The joints should always be framed of different substances; thus, one side or part should be of silver or brass, and the other of steel. The difference in the texture and pores of the two metals causes the parts to adhere less together, diminishes the wear, and promotes uniformity in their motion. The truth of the work is ascertained by the smoothness and equality of the motion at the joint, for all shake and irregularity is a certain sign of imperfection. The points should be of steel, so tempered as neither to be easily bent or blunted; not too fine and tapering, and yet meeting closely when the compasses are shut.

As an instrument of art, compasses are so well known that it would be superfluous to enumerate their various uses; suffice it then to say, that they are used to transfer small distances, measure given spaces, and describe arches and circles.

If an arch or circle is to be described obscurely, the steel points are best adapted to the purpose; if it is to be in ink or black lead, either the drawing pen, or crayon points are to be used.

To use a pair of compasses. Place the thumb and middle finger of the right hand in the opposite hollows in the shanks of the compasses, then press the compasses, and the legs will open a little way; this being done, push the innermost leg with the third finger, elevating at the same time the furthermost with the nail of the middle finger, till the compasses are sufficiently opened to receive the middle and third finger; they may then be extended at pleasure, by pushing the furthermost leg outwards with the middle, or pressing it inwards with the fore-finger. In describing circles or arches, set one foot of the compasses on the centre, and then roll the head of the compasses between the middle and fore-finger, the other point pressing at the same time upon the paper. They should be held as upright as possible, and care should be taken not to press forcibly upon them, but rather to let them act by their

own weight; the legs should never be so far extended as to form an obtuse angle with the paper or plane on which they are used.

The ink and crayon points have a joint just under that part which fits into the compasses; by this they may be always so placed as to be set nearly perpendicular to the paper; the end of the shank of the best compasses is framed so as to form a strong spring, to bind firmly the moveable points, and prevent them from shaking. This is found to be a more effectual method than that by a screw.

Two additional pieces are often applied to these compasses; these, by lengthening the leg, enable them to strike larger circles, or measure greater extents, than they would otherwise perform, and that without the inconveniences attending longer compasses. When compasses are furnished with this additional piece, the moveable leg has a joint, that it may be placed perpendicular to the paper.

The bow compasses are a small pair, usually with a point for ink; they are used to describe small arches or circles, which they do much more conveniently than large compasses, not only on account of their size, but also from the shape of the head, which rolls with great ease between the fingers.

Of the drawing pen and protracting pin. The pen part of this instrument is used to draw straight lines: it consists of two blades with steel points fixed to a handle; the blades are so bent that the ends of the steel points meet, and yet leave a sufficient cavity for the ink; the blades may be opened more or less by a screw, and, being properly set, will draw a line of any assigned thickness. One of the blades is framed with a joint, that the points may be separated and thus cleaned more conveniently. A small quantity only of ink should be put at one time into the drawing pen, and this should be placed in the cavity between the blades by a common pen or feeder; the drawing pen acts better if the pen by which the ink is inserted be made to pass through the blades. To use the drawing pen, first feed it with ink, then regulate it to the thickness of the required line by the screw. In drawing lines, incline the pen a small degree, taking care, however, that the edges of both the blades touch the paper, keeping the pen close to the rule, and in the same direction during the whole operation. The blades should always be wiped very clean before the pen is put away.

These directions are equally applicable to the ink point of the compasses, only observing, that when an arch or circle

is to be described of more than an inch radius, the point should be so bent that the blades of the pen may be nearly perpendicular to the paper, and both of them touch it at the same time.

The protracting pin is only a short piece of steel wire with a very fine point fixed at one end of the upper part of the handle of the drawing pen. It is used to mark the intersection of lines, or to set off divisions from the plotting scale and protractor.

OF THE SECTOR.

Amid the variety of mathematical instruments that have been contrived to facilitate the art of drawing, there is none so extensive in its use or of such general application as the *sector*. It is a universal scale, uniting, as it were, angles and parallel lines, the rule and the compass, which are the only means that geometry makes use of for measuring, whether in speculation or practice. The real inventor of this valuable instrument is unknown; yet of so much merit has the invention appeared, that it was claimed by *Galileo*, and disputed by nations.

This instrument derives its name from the tenth definition of the third book of *Euclid*, where he defines the sector of a circle. It is formed of two equal rules called legs; these legs are moveable about the centre of a joint, and will, consequently, by their different openings, represent every possible variety of plane angles. The distance of the extremities of these rules are the subtenses or chords, or the arches they describe.

Sectors are made of different sizes, but their length is usually denominated from the length of the legs when the sector is shut. Thus a sector of six inches when the legs are close together forms a rule of twelve inches when opened; and a foot sector is two feet long when opened to its greatest extent. In describing the lines usually placed on this instrument, I refer to those commonly laid down on the best six-inch brass sectors. But as the principles are the same in all, and the differences little more than in the number of subdivisions, it is to be presumed that no difficulty will occur in the application of what is here said to sectors of a larger radius.

The scales, or lines graduated upon the faces of the instrument, and which are to be used as *sectoral lines*, proceed from the centre, and are, 1. Two scales of equal parts, one on each leg, marked LIN. or L. Each of these scales, from the great extensiveness of its use, is called the *line of lines*. 2. Two lines of *chords*, marked CHO. or C. 3. Two lines of *secants*,

marked SEC. or S. A line of *polygons*, marked POL. Upon the other face the sectoral lines are, 1. Two lines of *sines* marked SIN. or S. 2. Two lines of *tangents*, marked TAN. 3. Between the lines of tangents and sines there is another line of tangents to a lesser radius, to supply the defect of the former, and extending from 45° to 75°.

Each pair of these lines, except the line of polygons, is so adjusted as to make equal angles at the centre, and consequently at whatever distance the sector be opened, the angles will be always respectively equal. That is, the distance between 10 and 10 on the line of lines will be equal to 60 and 60 on the line of chords, 90 and 90 on the line of sines, and 45 and 45 on the line of tangents.

Besides the sectoral scales, there are others on each face placed parallel to the outward edges, and used as those of the common plain scale. There are on the one face, 1. A line of inches. 2. A line of latitudes. 3. A line of hours. 4. A line of inclination of meridians. 5. A line of chords. On the other face, three logarithmic scales, namely, one of numbers, one of sines, and one of tangents: these are used when the sector is fully opened, the legs forming one line.

To read and estimate the divisions on the sectoral lines. The value of the divisions on most of the lines is determined by the figures adjacent to them; these proceed by tens, which constitute the divisions of the first order, and are numbered accordingly; but the value of the divisions on the line of lines, that are distinguished by figures, is entirely arbitrary, and may represent any value that is given to them; hence the figures 1, 2, 3, 4, &c. may denote either 10, 20, 30, 40, or 100, 200, 300, 400, and so on.

The line of lines is divided into ten equal parts, numbered 1, 2, 3, to 10; these may be called divisions of the first order; each of these is again subdivided into 10 other equal parts, which may be called divisions of the second order; and each of these is divided into two equal parts, forming divisions of the third order.

The divisions on all the scales are contained between four parallel lines: those of the first order extend to the most distant, those of the third to the least, those of the second to the intermediate parallel.

When the whole line of lines represents 100, the divisions of the first order, or those to which the figures are annexed, represent tens; those of the second order, units; those of the third order, the halves of these units. If the whole line repre-

sents ten, then the divisions of the first order are units; those of the second, tenths; and the third, twentieths.

In the line of tangents, the divisions to which the numbers are affixed are the degrees expressed by those numbers. Every fifth degree is denoted by a line somewhat longer than the rest; between every number and each fifth degree there ar four divisions longer than the intermediate adjacent ones; these are whole degrees; the shorter ones, or those of the third order, are 30 minutes.

From the centre to 60 degrees *the line of sines* is divided like the line of tangents; from 60 to 70 it is divided only to every degree; from 70 to 80 to every two degrees; from 80 to 90 the division must be estimated by the eye.

The divisions on *the line of chords* are to be estimated in the same manner as the tangents.

The *lesser line of tangents* is graduated every two degrees from 45 to 50; but from 50 to 60 to every degree; from 60 to the end to half-degrees.

The *line of secants* from 0 to 10 is to be estimated by the eye; from 20 to 50 it is divided to every two degrees; from 50 to 60 to every degree; and from 60 to the end to every half-degree.

The solution of questions on the sector is said to be *simple* when the work is begun and ended on the same line; *compound* when the operation begins on one line and is finished on the other.

The operation varies also by the manner in which the compasses are applied to the sector. If a measure be taken on any of the sectoral lines beginning at the centre, it is called a *lateral distance.* But if the measure be taken from any point in one line to its corresponding point on the line of the same denomination on the other leg, it is called *a transverse* or *parallel distance.*

The divisions of each sectoral line are bounded by three parallel lines; the innermost of these is that on which the points of the compasses are to be placed, because this alone is the line which goes to the centre, and is alone, therefore, the sectoral line.

We shall now proceed to give a few general instances of the manner of operating with the sector.

Multiplication by the line of lines. Make the lateral distance of one of the factors the parallel distance of 10; then the parallel distance of the other factor is the product.

Example. Multiply 5 by 6: extend the compasses from the

centre of the sector to 5 on the primary divisions, and open the sector till this distance become the parallel distance from 10 to 10 on the same divisions; then the parallel distance from 6 to 6, extended from the centre of the sector, shall reach to 3, which is now to be reckoned 30. At the same opening of the sector, the parallel distance of 7 shall reach from the centre to 35, that of 8 shall reach from the centre to 40, &c.

Division by the line of lines. Make the lateral distance of the dividend the parallel distance of the divisor, the parallel distance of 10 is the quotient. Thus, to divide 30 by 5, make the lateral distance of 30, viz. 3 on the primary divisions, the parallel distance of 5 of the same divisions; then the parallel distance of 10, extended from the centre, shall reach to 6.

Proportion by the line of lines. Make the lateral distance of the second term the parallel distance of the first term; the parallel distance of the third term is the fourth proportional.

Example. To find a fourth proportional to 8, 4, and 6, take the lateral distance of 4, and make it the parallel distance of 8, then the parallel distance of 6, extended from the centre, shall reach to the fourth proportional 3.

In the same manner a third proportional is found to two numbers. Thus, to find a third proportional to 8 and 4, the sector remaining as in the former example, the parallel distance of 4, extended from the centre, shall reach to the third proportional 2. In all these cases, if the number to be made a parallel distance be too great for the sector, some aliquot part of it is to be taken, and the answer multiplied by the number by which the first number was divided. Thus, if it were required to find a fourth proportional to 4, 8, and 6, because the lateral distance of the second term 8 cannot be made the parallel distance of the first term 4, take the lateral distance of 4, viz. the half of 8, and make it the parallel distance of the first term 4; then the parallel distance of the third term 6 shall reach from the centre to 6, viz. the half of 12. Any other aliquot part of a number may be used in the same way. In like manner, if the number proposed be too small to be made the parallel distance, it may be multiplied by some number, and the answer is to be divided by the same number.

To protract angles by the line of chords. Case 1. When the given degrees are under 60. 1. With any radius on a centre, describe the arch. 2. Make the same radius a transverse distance between 60 and 60 on the same line of chords. 3. Take out the transverse distance of the given degrees, and lay this on the arch, which will mark out the angular distance required.

Case 2. When the given degrees are more than 60. 1. Open the sector, and describe the arch as before. 2. Take $\frac{1}{2}$ or $\frac{1}{3}$ of the given degrees, and take the transverse distance of this $\frac{1}{2}$ or $\frac{1}{3}$, and lay it off twice if the degrees were halved, three times if the third was used as a transverse distance.

Case 3. When the required angle is less than 6 degrees; suppose 3. 1. Open the sector to the given radius, and describe the arch as before. 2. Set off the radius. 3. Set off the chord of 57 degrees backwards, which will give the arc of three degrees.

Given the radius of a circle (suppose equal to two inches), required the sine and tangent of 28° 30′ *to that radius.*

Solution.—Open the sector so that the transverse distance of 90 and 90 on the sines, or of 45 and 45 on the tangents, may be equal to the given radius, viz. two inches; then will the transverse distance of 28° 30′, taken from the sines, be the length of that sine to the given radius; or if taken from the tangents, will be the length of that tangent to the given radius.

But if the secant of 28° 30′ *was required?*

Make the given radius, two inches, a transverse distance to 0 and 0 at the beginning of the line of secants; and then take the transverse distance of the degrees wanted, viz. 28° 30′.

A tangent greater than 45° (*suppose* 60°) *is found thus.*

Make the given radius, suppose two inches, a transverse distance to 45 and 45 at the beginning of the scale of upper tangents; and then the required number 60° may be taken from this scale.

Given the length of the sine, tangent, or secant of any degrees, to find the length of the radius to that sine, tangent, or secant.

Solution.—Make the given length a transverse distance to its given degrees on its respective scale: then,

In the sines. The transverse distance of 90 and 90 will be the radius sought.

In the lower tangents. The transverse distance of 45 and 45, near the end of the sector, will be the radius sought.

In the upper tangents. The transverse distance of 45 and 45, taken towards the centre of the sector on the line of upper tangents, will be the centre sought.

In the secant. The transverse distance of 0 and 0, or the beginning of the secants, near the centre of the sector, will be the radius sought.

Given the radius and any line representing a sine, tangent, or secant, to find the degrees corresponding to that line.

Solution.—Set the sector to the given radius, according as a sine, or tangent, or secant is concerned.

Take the given line between the compasses; apply the two feet transversely to the scale concerned, and slide the feet along till they both rest on like divisions on both legs; then will those divisions show the degrees and parts corresponding to the given line.

To find the length of a versed sine to a given number of degrees, and a given radius.

Make the transverse distance of 90 and 90 on the sines equal to the given radius.

Take the transverse distance of the sine complement of the given degrees.

If the given degrees are less than 90, the difference between the sine complement and the radius gives the versed sine.

If the given degrees are more than 90, the sum of the sine complement and the radius gives the versed sine.

To open the legs of the sector so that the corresponding double scales of lines chords, sines, and tangents may make each a right angle.

On the lines, make the lateral distance 10 a distance between 8 on one leg and 6 on the other leg.

On the sines, make the lateral distance 90 a transverse distance from 45 to 45; or from 40 to 50; or from 30 to 60; or from the sine of any degrees to their complement.

Or on the sines, make the lateral distance of 45 a transverse distance between 30 and 30.

OF THE PLAIN SCALE.

The divisions laid down on the plain scale are of two kinds, the one having more immediate relation to the circle and its properties, the other being merely concerned with dividing straight lines.

Though arches of a circle are the most natural measure of an angle, yet in many cases right lines are substituted, as being more convenient; for the comparison of one right line with another is more natural and easy than the comparison of a right line with a curve: hence it is usual to measure the quantities of angles, not by the arch itself, which is described on the angular point, but by certain lines described about that arch.

The lines laid down on the plain scales for the measuring of angles, or the protracting scales, are, 1. A line of *chords* marked CHO. 2. A line of *sines* marked SIN., of *tangents* marked TAN., of *semitangents* marked ST., and of *secants* marked SEC.; this last is often upon the same line as the sines, because its gradations do not begin till the sines end.

There are two other scales, namely, the *rhumbs* marked RU, and *longitudes* marked LON. Scales of latitude and hours are sometimes put upon the plain scale; but as dialling is now but seldom studied, they are only made to order.

The divisions used for measuring straight lines are called *scales of equal parts*, and are of various lengths for the convenience of delineating any figure of a larger or smaller size, according to the fancy or purposes of the draughtsman. They are, indeed, nothing more than a measure in miniature for laying down upon paper, &c. any known measure, as chains, yards, feet, &c., each part on the scale answering to one foot, one yard, &c., and the plan will be larger or smaller as the scale contains a smaller or a greater number of parts in an inch. Hence a variety of scales is useful to lay down lines of any required length, and of a convenient proportion with respect to the size of the drawing. If none of the scales happen to suit the purpose, recourse should be had to the *line of lines* on the sector; for, by the different openings of that instrument, a line of any length may be divided into as many equal parts as any person chooses.

Scales of equal parts are divided into two kinds, the one simply, the other diagonally divided.

Six of the simply divided scales are generally placed one above another upon the same rule; they are divided into as many equal parts as the length of the rule will admit of; the numbers placed on the right-hand show how many parts in an inch each scale is divided into. The upper scale is sometimes shortened for the sake of introducing another, called the line of chords.

The first of the larger or primary divisions on every scale is subdivided into ten equal parts, which small parts are those which give a name to the scale: thus it is called a scale of 20, when 20 of these divisions are equal to one inch. If, therefore, these less divisions be taken as units, and each represents one league, one mile, one chain, or one yard, &c., then will the larger divisions be so many tens; but if the subdivisions are supposed to be tens, the larger divisions will be hundreds.

To illustrate this, suppose it were required to set off from either of the scales of equal parts $\frac{4}{10}$, 36, or 360 parts, either miles or leagues. Set one foot of your compasses on 3, among the larger or primary divisions, and open the other point till it falls on the sixth subdivision, reckoning backwards or towards the left hand. Then will this extent represent $\frac{3}{10}$, 36, or 360 miles or leagues, &c. and bear the same proportion in the plan as the line measured does to the thing represented.

To adapt these scales to feet and inches, the first primary division is often duodecimally divided by the upper line; therefore, to lay down any number of feet and inches, as, for instance, 8 feet 8 inches, extend the compasses from 8 of the larger to 8 of the upper small ones, and that distance laid down on the plan will represent 8 feet 8 inches.

Of the scale of equal parts diagonally divided. The use of this scale is the same as those already described. But by it a plane may be more accurately divided than by the former; for any one of the larger divisions may by this be subdivided into 100 equal parts; and, therefore, if the scale contains 10 of the larger divisions, any number under 1000 may be laid down with accuracy.

The diagonal scale is seldom placed on the same side of the rule with the other plotting scale. The first division of the diagonal scale, if it be a foot long, is generally an inch divided into 100 equal parts, and at the opposite there is usually half an inch divided into 100 equal parts. If the scale be six inches long, one end has commonly half an inch, the other a quarter of an inch, subdivided into 100 equal parts.

The nature of this scale will be better understood by considering its construction. For this purpose,

First. Draw eleven parallel lines at equal distances; divide the upper of these lines into such a number of equal parts as the scale to be expressed is intended to contain; from each of these divisions draw perpendicular lines through the eleven parallels.

Secondly. Subdivide the first of these divisions into ten equal parts, both in the upper and lower lines.

Thirdly. Subdivide again each of these subdivisions, by drawing diagonal lines from the 10th below to the 9th above; from the 8th below to the 7th above; and so on, till from the first below to the 0 above; by these lines each of the small divisions is divided into ten parts, and consequently the whole first space into 100 equal parts; for as each of the subdivisions is one-tenth part of the whole first space or division, so each parallel above it is one-tenth of such subdivision, and consequently, one-hundredth part of the whole first space; and if there be ten of the larger divisions, one thousandth part of the whole space.

If, therefore, the larger divisions be accounted as units, the first subdivisions will be tenth parts of a unit, and the second, marked by the diagonal upon the parallels, hundredth parts of the unit. But if we suppose the larger divisions to be tens, the first subdivisions will be units and the second tenths. If

the larger are hundreds, then will the first be tens and the second units.

The numbers, therefore, 576, 57,6, 5,76, are all expressible by the same extent of the compasses: thus, setting one foot in the number 5 of the larger divisions, extend the other along the sixth parallel to the seventh diagonal. For, if the five larger divisions be taken for 500, seven of the first subdivisions will be 70, which upon the sixth parallel, taking in six of the second subdivisions for units, makes the whole number 576. Or, if the five larger divisions be taken for five tens, or 50, seven of the first subdivisions will be seven units, and the six second subdivisions upon the sixth parallel will be six tenths of a unit. Lastly, if the five larger divisions be only esteemed as five units, then will the seven first subdivisions be seven tenths, and the six second subdivisions be the six hundredth parts of a unit.

Of the line of chords. This line is used to set off an angle from a given point in any right line, or to measure the quantity of an angle already laid down.

Thus, to draw a line that shall make with another line an angle containing a given number of degrees, suppose 40 degrees.

Open your compasses to the extent of 60 degrees upon the line of chords (which is always equal to the radius of the circle of projection), and setting one foot in the angular point, with that extent describe an arch; then taking the extent of 40 degrees from the said chord line, set it off from the given line on the arch described; a right line drawn from the given point through the point marked upon the arch will form the required angle.

The degrees contained in an angle already laid down are found nearly in the same manner. For instance, to measure an angle: from the centre describe an arch with the chord of 60 degrees, and the length of the arch contained between the lines measured on the line of chords will give the number of degrees contained in the angle.

If the number of degrees are more than 90, they must be measured upon the chords at twice: thus, if 120 degrees were to be practised, 60 may be taken from the chords, and those degrees be laid off twice upon the arch. Degrees taken from the chords are always to be counted from the beginning of the scale.

Of the rhumb line. This is, in fact, a line of chords constructed to a quadrant divided into eight parts or points of the compass, in order to facilitate the work of the navigator in laying down a ship's course.

Of the line of longitudes. The line of longitudes is a line divided into sixty unequal parts, and so applied to the line of chords as to show, by inspection, the number of equatorial miles contained in a degree on any parallel of latitude. The graduated line of chords is necessary, in order to show the latitudes; the line of longitude shows the quantity of a degree on each parallel in sixtieth parts of an equatorial degree, that is, miles.

The lines of tangents, semitangents and secants serve to find the centres and poles of projected circles in the stereographical projection of the sphere.

The line of sines is principally used for the orthographic projection of the sphere.

The lines of latitudes and hours are used conjointly, and serve very readily to mark the hour lines in the construction of dials: they are generally on the most complete sorts of scales and sectors; for the uses of which see treatises on dialling.

OF THE PROTRACTOR.

This is an instrument used to protract or lay down an angle containing any number of degrees, or to find how many degrees are contained in any given angle. There are two kinds put into cases of mathematical drawing instruments; one in the form of a semicircle, the other in the form of a parallelogram. The circle is undoubtedly the only natural measure of angles; when a straight line is therefore used the divisions thereon are derived from a circle or its properties, and the straight line is made use of for some relative convenience: it is thus the parallelogram is often used as a protractor instead of the semicircle, because it is in some cases more convenient, and that other scales, &c. may be placed upon it.

The semicircular protractor is divided into 180 equal parts or degrees, which are numbered at every tenth degree each way, for the conveniency of reckoning either from the right towards the left, or from the left towards the right; or the more easily to lay down an angle from either end of the line, beginning at each end with 10, 20, &c. and proceeding to 180 degrees. The edge is the diameter of the semicircle, and the mark in the middle points out the centre, in a *protractor in the form of a parallelogram:* the divisions are, as in the semicircular one, numbered both ways; the blank side represents the diameter of a circle. The side of the protractor to be applied to the paper is made flat, and that whereon the degrees are marked is chamfered or sloped away to the edge, that an angle

may be more easily measured, and the divisions set off with greater exactness.

Application of the protractor to use. 1. *A number of degrees being given, to protract or lay down an angle whose measure shall be equal thereto.*

Thus, to lay down an angle of 60 degrees from the point of a line, apply the diameter of the protractor to the line, so that the centre thereof may coincide exactly with the extremity; then, with a protracting pin make a fine dot against 60 upon the limb of the protractor; now remove the protractor, and draw a line from the extremity through that point, and the angle contains the given number of degrees.

2. *To find the number of degrees contained in a given angle.*

Place the centre of the protractor upon the angular point, and the fiducial edge or diameter exactly upon the line; then the degree upon the limb that is cut by the line will be the measure of the given angle, which, in the present instance is found to be 60 degrees.

3. *From a given point in a line to erect a perpendicular to that line.*

Apply the protractor to the line, so that the centre may coincide with the given point, and the division marked 90 may be cut by the line; then a line drawn against the diameter of the protractor will be the perpendicular required.

OF PARALLEL RULES.

Parallel lines occur so continually in every species of mathematical drawing, that it is no wonder so many instruments have been contrived to delineate them with more expedition than could be effected by the general geometrical methods. For this purpose *rules* of various constructions have been made, and particularly recommended by their inventors; their use, however, is so apparent as to need no explanation.

GUNTER'S SCALE.

The scale generally used is a ruler two feet in length, having drawn upon it equal parts, chords, sines, tangents, secants, &c. These are contained on one side of the scale, and the other side contains the logarithms of these numbers. *Mr. Edmund Gunter* was the first who applied the logarithms of numbers and of sines and tangents to straight lines drawn on a scale or ruler, with which proportions in common numbers and trigonometry may be solved by the application of a pair of compasses only. The method is founded on this property,

That the logarithms of the terms of equal ratios are equidifferent. This was called Gunter's Proportion and Gunter's Line; hence the scale is generally called the Gunter.

Of the Logarithmical Lines on Gunter's Scale.

The logarithmical lines on Gunter's Scale are the eight following:

S. Rhumb, or fine rhumbs, is a line containing the logarithms of the natural sines of every point and quarter point of the compass, numbered from a brass pin on the right-hand towards the left with 8, 7, 6, 5, 4, 3, 2, 1.

T. Rhumb, or tangent rhumbs, also corresponds to the logarithm of the tangent of every point and quarter point of the compass. This line is numbered from near the middle of the scale with 1, 2, 3, 4, towards the right-hand, and back again with the numbers 5, 6, 7, from the right-hand towards the left. To take off any number of points below 4, we must begin at 1 and count towards the right-hand; but to take off any number of points above 4, we must begin at 4 and count towards the left-hand.

Numbers, on the line of numbers, is numbered from the left-hand of the scale towards the right, with 1, 2, 3, 4, 5, 6, 7, 8, 9, 1, which stands exactly in the middle of the scale; the numbers then go on 2, 3, 4, 5, 6, 7, 8, 9, 10, which stands at the right-hand end of the scale. These two equal parts of the scale are divided equally, the distance between the first or left-hand 1 and the first 2, 3, 4, &c. is exactly equal to the distance between the middle 1 and numbers 2, 3, 4, &c. which follow it. The subdivisions of these scales are likewise similar, viz. they are each one-tenth of the primary divisions, and are distinguished by lines of about half the length of the primary divisions.

These subdivisions are again divided into ten parts, where room will permit; and where that is not the case the units must be estimated or guessed at by the eye, which is easily done by a little practice.

The primary divisions on the second part of the scale are estimated according to the value set upon the unit on the left-hand of the scale: if you call it one, then the first 1, 2, 3, &c. stand for 1, 2, 3, &c.; the middle 1 is 10, and the 2, 3, 4, &c. following stand for 20, 30, 40, &c.; and the 10 at the right-hand is 100. If the first 1 stand for 10, the first 2, 3, 4, &c. must be counted 20, 30, 40, &c.; the middle 1 will be 100, and the second 2, 3, 4, 5, &c. will stand for 200, 300, 400, 500, &c.; and the 10 at the right for 1000.

If you consider the first 1 as $\frac{1}{10}$ of a unit, the 2, 3, 4, &c.

following will be $\frac{2}{10}$, $\frac{3}{10}$, $\frac{4}{10}$, &c.; the middle 1 will stand for a unit, and the 2, 3, 4, &c. following will stand for 2, 3, 4, &c.; also, the division at the right-hand end of the scale will stand for 10. The intermediate small divisions must be estimated according to the value set upon the primary ones.

Sine. The line of sines is numbered from the left-hand of the scale towards the right, 1, 2, 3, 4, 5, &c. to 10; then 20, 30, 40, &c. to 90, where it terminates just opposite 10 on the line of numbers.

Versed sine. This line is placed immediately under the line of sines, and numbered in a contrary direction, viz. from the right-hand towards the left, 10, 20, 30, 40, 50, to about 160; the small divisions are here to be estimated according to the number of them to a degree. By comparing the line of versed sines with the line of sines, it will appear that the versed sines do not belong to the arches with which they are marked, but are the half versed sines of their supplements. Thus, what is marked the versed sine of 90 is only half the versed sine of 90, the versed sine of 120° is half the versed sine of 60°, and the versed sine marked 100° is half the versed sine of 80°, &c.

The versed sines are numbered in this manner to render them more commodious in the solution of trigonometrical and astronomical problems.

Tangents. The line of tangents begins at the left-hand, and is numbered 1, 2, 3, &c. to 10, then 20, 30, 45, where there is a little brass pin just under 90 in the line of sines, because the sine of 90° is equal to the tangent of 45°. It is numbered from 45° towards the left-hand 50, 60, 70, 80, &c. The tangents of arches above 45° are therefore counted backward on the line, and are found at the same points of the line as the tangents of their complements.

Thus the division at 40 represents both 40 and 50, the division at 30 serves for 30 and 60, &c.

Meridional Parts. This line stands immediately above a line of equal parts, marked *Equal Pt.*, with which it must always be compared when used. The line of equal parts is marked from the right-hand to the left with 0, 10, 20, 30, &c.; each of these large divisions represents 10 degrees of the equator, or 600 miles. The first of these divisions is sometimes divided into 40 equal parts, each representing 15 minutes or miles.

The extent from the brass pin on the scale of meridional parts to any *division* on that scale, applied to the line of equal parts, will give (in degrees) the meridional parts answering to

the latitude of that *division*. Or the extent from any *division* to another on the line of meridional parts, applied to the line of equal parts, will give the meridional difference of latitude between the two places denoted by the *divisions*. These degrees are reduced to leagues by multiplying by 20, or to miles by multiplying by 60.

The use of the Logarithmical Lines on Gunter's Scale.

By these lines and a pair of compasses all the problems of trigonometry, &c. may be solved.

These problems are all solved by proportion. Now, in natural numbers the quotient of the first term by the second is equal to the quotient of the third by the fourth: therefore, logarithmically speaking, the difference between the first and second term is equal to the difference between the third and fourth; consequently, on the lines on the scale the distance between the first and second term will be equal to the distance between the third and fourth. And for a similar reason, because four proportional quantities are alternately proportional, the distance between the first and third terms will be equal to the distance between the second and fourth. Hence the following

General Rule.

The extent of the compasses from the first term to the second will reach, in this same direction, from the third to the fourth term. Or, the extent of the compasses from the first term to the third will reach, in the same direction, from the second to the fourth.

By the same direction in the foregoing rule is meant, that if the second term lie on the right-hand of the first the fourth will lie on the right-hand of the third, and the contrary. This is true, except the two first or two last terms of the proportion are on the line of tangents, and neither of them under 45°; in this case, the extent on the tangents is to be made in a contrary direction: for had the tangents above 45° been laid down in their proper direction, they would have extended beyond the length of the scale towards the right-hand; they are therefore, as it were, folded back upon the tangents below 45°, and consequently lie in a direction contrary to their proper and natural order.

If the two last terms of a proportion be on the line of tangents, and one of them greater and the other less than 45°, the extent from the first term to the second will reach from the third beyond the scale. To remedy this inconvenience, apply the extent between the two first terms from 45° backward upon the line of tangents, and keep the left-hand point of the

compasses where it falls; bring the right-hand point from 45° to the third term of the proportion; this extent now in the compasses applied from 45° backward will reach to the fourth term, or the tangent required. For, had the line of tangents been continued forward beyond 45°, the divisions would have fallen above 45° forward, in the same manner as they fall under 45° backward.

SECTION V.

TRIGONOMETRY.

The word *Trigonometry* signifies the measuring of triangles. But under this name is generally comprehended the art of determining the positions and dimensions of the several unknown parts of extension, by means of some parts which are already known. If we conceive the different points which may be represented in any space to be joined together by right lines, there are three things offered for our consideration; 1, the length of these lines; 2, the angles which they form with one another; 3, the angles formed by the planes in which these lines are drawn, or are supposed to be traced. On the comparison of these three objects depends the solution of all questions that can be proposed concerning the measure of extension and its parts; and the art of determining all these things from the knowledge of some of them is reduced to the solution of these two general questions.

1. Knowing three of the six parts, the sides and angles, which constitute a rectilineal triangle, to find the other three.

2. Knowing three of the six parts which compose a spherical triangle, that is, a triangle formed on the surface of a sphere by three arches of circles which have their centre in the centre of the same sphere, to find the other three.

The first question is the object of what is called Plane Trigonometry, because the six parts considered here are in the same plane: it is also denominated Rectilineal Trigonometry. The second question belongs to Spherical Trigonometry, wherein the six parts are considered in different planes. But the only object here is to explain the solutions of the former question, viz.

PLANE TRIGONOMETRY.

Plane Trigonometry is that branch of Geometry which

teaches how to determine or calculate three of the six parts of a rectilineal triangle, by having the other three parts given or known. It is usually divided into Right-angled and Oblique-angled Trigonometry, according as it is applied to the mensuration of right or oblique-angled triangles.

In every triangle or case in trigonometry three of the parts must be given, and one of these parts at least must be a side; because, if the three angles only were given, it is obvious that all similar triangles would answer the question.

RIGHT-ANGLED PLANE TRIGONOMETRY.

PL. 5. *fig.* 1.

1. In every right-angled plane triangle ABC, if the hypothenuse AC be made the radius, and with it a circle or an arc of one be described from each end, it is plain (from def. 20), that BC is the sine of the angle A, and AB is the sine of the angle C; that is, the legs are the sines of their opposite angles.*

* The sine and co-sine of any number of degrees and minutes is found by the series (which is given and illustrated in page 49, Ryan's *Differential and Integral Calculus*)

$a-\frac{a^3}{2\cdot3}+\frac{a^5}{2\cdot3\cdot4\cdot5}-\frac{a^7}{2\cdot3\cdot4\cdot5\cdot6\cdot7}$, &c. for its sine, and its co-sine by $1-\frac{a^2}{2}+\frac{a^4}{2\cdot3\cdot4}-\frac{a^6}{2\cdot3\cdot4\cdot5\cdot6}+\frac{a^8}{2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8}$, &c.

In which series the value of a is found thus: as the number of degrees or minutes in the whole semicircle is to the degrees or minutes in the arc proposed, so is 3.14159, &c. to the length of the said arc, which is the value of a. For example, let it be required to find the sine of one minute; then, as 10800 (the minutes in 180 degrees) : 1 : : 3.14159, &c. : .000290888208665= the length of an arc of one minute, which is the value of a in this case, and $\frac{a^3}{2\cdot3}(=\frac{a^3}{6})$=.000000000004102, &c. Consequently, .000290888208665—.000000000004102=.000290888204563= the required sine of one minute.

Again, let it be required to find the sine and co-sine of five degrees, each true to seven places of decimals. Here, .0002908882, the length of an arc of one minute (found above), being multiplied by 300, the number of minutes in 5 degrees, the product .08726646 is the length of an arc of 5 degrees; therefore in this case we have

$$\begin{aligned} a &= \quad .08726646 \\ -\frac{a^3}{6} &= -.00011076 \\ +\frac{a^5}{120} &= +.00000004 \\ \text{\&c.} &\qquad \text{\&c.} \end{aligned}$$

Consequently, .08715574 = the sine of 5 degrees. Also, $\frac{a^2}{2}$=.00380771, and $\frac{a^4}{24}$=.00000241; hence 1—.00380771 + .00000241=.9961947= the co-sine of 5 degrees.

Fig. 2.

2. If one leg AB be made the radius, and with it on the point A an arc be described, then BC is the tangent and AC is the secant of the angle A, by def. 22 and 25.

Fig. 3.

3. If BC be made the radius, and an arc be described with it on the point C, then is AB the tangent and AC is the secant of the angle C, as before.

Because the sine, tangent, or secant of any given arc in one circle is to the sine, tangent, or secant of a like arc (or to one of the like number of degrees) in another circle, as the radius of the one is to the radius of the other; therefore the sine, tangent, or secant of any arc is proportional to the sine, tangent, or secant of a like arc, as the radius of the given arc is to 10.000000, the radius from whence the logarithmic sines, tangents, and secants in most tables are calculated; that is,

If AC be made the radius, the sines of the angles A and C, described by the radius AC, will be proportional to the sines of the like arcs or angles in the circle that the tables now mentioned were calculated for. So if BC was required, having the angles and AB given, it will be,

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After the same manner the sine and co-sine of any other arc may be derived; but the greater the arc is the slower the series will converge, and therefore a greater number of terms must be taken to bring out the conclusion to the same degree of exactness.

Or, having found the sine, the co-sine will be found from it (by theo. 14), the co-sine CL (plate 1, fig. 8) $= \sqrt{CH^2, HL^2}$, or $c = \sqrt{1 - s^2}$.

For other methods of constructing the canon of sines and co-sines, the reader is referred to Hutton's Mathematics, Simpson's Algebra, &c.

The sines and co-sines being known or found by the foregoing method, the tangents and secants will be easily found from the principle of similar triangles, in the following manner:

In plate 1, fig. 8, where of the arc BH, HL is the sine, CL or FH the co-sine, BK the tangent, CK the secant, DI the co-tangent, and CI the co-secant, the radius being CH, or CB, or CD, the three similar triangles CLH, CBK, CDI give the following proportion (by theo. 14):

1. $CL : LH :: CB : BK$; whence the tangent is known, being a fourth proportional to the co-sine, sine, and radius.

2. $CL : CH :: CB : CK$; whence the secant is known, being a third proportional to the co-sine and radius.

3. $HL : LC :: CD : DI$; whence the co-tangent is known, being a fourth proportional to the sine, co-sine, and radius.

4. $HL : HC :: CD : CI$; whence the co-secant is known, being a third proportional to the sine and radius.

As for the logarithms, sines, tangents, and secants in the tables, they are only the logarithms of the natural sines, tangents, and secants calculated as above.

Fig. 1.

As $S.C : AB :: S.A : BC$.

That is, as the sine of the angle C in the tables is to the length of AB (or sine of the angle C in a circle whose radius is AC), so is the sine of the angle A in the tables to the length of BC (or sine of the same angle in the circle whose radius is AC).

In like manner the tangents and secants represented by making either leg the radius will be proportional to the tangents and secants of a like arc, as the radius of the given arc is to 10.000000, the radius of the tables aforesaid.

Hence it is plain, that if the name of each side of the triangle be placed thereon, a proportion will arise to answer the same end as before: thus, if AC be made the radius, let the word radius be written thereon; and as BC and AB are the sines of their opposite angles, upon the first let $S.A$, or sine of the angle A, and on the other let $S.C$, or sine of the angle C, be written. Then,

When a side is required, it may be obtained by this proportion, viz.

As the name of the side given
is to the side given,
So is the name of the side required
to the side required.

Thus, if the angles A and C and the hypothenuse AC were given, to find the sides; the proportion will be

Fig. 1.

1. $R : AC :: S.A : BC$.

That is, as radius is to AC, so is the sine of the angle A to BC. And,

2. $R : AC :: S.C : AB$.

That is, as radius is to AC, so is the sine of the angle C to AB.

When an angle is required we use this proportion, viz.

As the side that is made the radius
is to radius,
So is the other given side
to its name.

Thus, if the legs were given, to find the angle A, and if AB be made the radius, it will be

Fig. 2.

$AB : R :: BC : T.A$.

That is, as AB is to radius, so is BC to the tangent of the angle A.

After the same manner, the sides or angles of all right-angled plane triangles may be found, from their proper data.

We here, in plate 4, give all the proportion requisite for the solution of the six cases in right-angled trigonometry; making every side possible the radius.

In the following triangles this mark — in an angle denotes it to be known, or the quantity of degrees it contains to be given; and this mark ′ on a side denotes its length to be given in feet, yards, perches, or miles, &c. and this mark °, either in an angle or on a side, denotes the angle or side to be required.

From these propositions it may be observed, that to find a side, when the angles and one side are given, any side may be made the radius; and to find an angle, one of the given sides must be made the radius. So that in the 1st, 2d, and 3d cases any side, as well required as given, may be made the radius, and in the first statings of the 4th, 5th, and 6th cases, a given side only is made the radius.

RIGHT-ANGLED TRIANGLES.

Case I.

The angles and hypothenuse given, to find the base and perpendicular.

Pl. 5. *fig.* 4.

In the right-angled triangle ABC, suppose the angle $A=46° 30'$; and consequently the angle $C=43° 30'$ (by cor. 2, theo. 5); and AC 250 parts (as feet, yards, miles, &c.); required the sides AB and BC.

1st. *By Construction.*

Make an angle of 46° 30′ in blank lines (by prob. 16, geom.), as CAB; lay 250, which is the given hypothenuse, from a scale of equal parts, from A to C; from C let fall the perpendicular BC (by prob. 7, geom.), and that will constitute the triangle ABC. Measure the lines BC and AB from the same scale of equal parts that AC was taken from, and you have the answer.*

* It is proper to observe, that constructions, though perfectly correct in theory, would give only a moderate approximation in practice, on account of the imperfection of the instruments required in constructing them; they are called *graphic methods*. Trigonometrical methods, on the contrary, being independent of all mechanical operation, give solutions with the utmost accuracy: they are founded upon the properties of lines called *sines, co-sines, tangents*, &c., which furnish a very simple mode of expressing the relations that subsist between the sides and angles of triangles.

2d. *By Calculation.*

1. *Making AC the radius*, the required sides are found by these propositions, as in plate 4, case 1.

$R : AC :: S.A : BC.$
$R : AC :: S.C : AB.$

That is, as radius	$=90°$	10.000000
is to AC,	$=250$	2.397940
So is the sine of A	$=46° 30'$*	9.860562
to BC,	$=181.4$	2.258502
As radius	$=90°$	10.000000
is to AC,	$=250$	2.397940
So is the sine of C	$=43° 30'$	9.837812
to AB,	$=172.1$	2.235752

If from the sum of the second and third logs. that of the first be taken, the number will be the log. of the fourth; the number answering to which will be the thing required; but when the first log. is radius, or 10.000000, reject the first figure of the sum of the other two logs. (which is the same thing as to subtract 10.000000), and that will be the log. of the thing required.

2. *Making AB the radius.*

Secant $A : AC :: R : AB.$
Secant $A : AC :: T.A : BC.$

That is, as the secant of A	$=46° 30'$	10.162188
is to AC,	$=250$	2.397940
So is the radius	$=90°$	10.000000
		12.397940
to AB,	$=172.1$	2.235762
As the secant of A	$=46° 30'$	10.162188
is to AC,	$=250$	2.397940
So is the tangent of A	$=46° 30'$	10.022750
		12.420690
to BC,	$=181.34$	2.258502

* For finding the logarithmic sine, co-sine, &c. of any number of degrees and minutes in the table, also the degrees, minutes, &c. of any logarithmic sine, co-sine, &c., the reader is referred to table 2, at the end of this treatise.

3. *Making BC the radius.*

Sec. C : AC : : R : BC.

Sec. C : AC : : $T.C$: AB.

That is, as the secant of C	=43° 30′	10.139438
is to AC,	=250	2.397940
So is the radius	=90°	10.000000
		12.397940
to BC,	=181.34	2.258502
As the secant of C	=43° 30′	10.139438
is to AC,	=250	2.397940
So is the tangent of C	=43° 30′	9.977250
		12.375190
to AB,	=172.1	2.235752

Or, having found one side, the other may be obtained by cor. 2, theo. 14, sect. 4.

3d. *By Gunter's Scale.*

The first and third terms in the foregoing proportions being of a like nature, and those of the second and fourth being also like to each other; and the proportions being direct ones; it follows, that if the third term be greater or less than the first, the fourth term will be also greater or less than the second: therefore the extent in your compasses from the first to the third term will reach from the second to the fourth.

Thus, to extend the first of the foregoing proportions;

1. Extend from 90° to 46° 30′, on the line of sines; that distance will reach from 250, on the line of numbers, to 181, for BC.

2. Extend from 90° to 43° 30′, on the line of sines; that distance will reach from 250, on the line of numbers, to 172, for AB.

If the first extent be from a greater to a less number; when you apply one point of the compasses to the second term, the other must be turned to a less; and the contrary.

By def. 20, sect. 4. The sine of 90° is equal to the radius; and the tangent of 45° is also equal to the radius; because if one angle of a right-angled triangle be 45°, the other will be also 45°; and thence (by the lemma preceding theo. 7, sect. 4) the tangent of 45° is equal to the radius: for this reason the line of numbers of 10.000000, the sine of 90°, and tangent of 45°, being all equal, terminate at the same end of the scale.

The first two statings of this case answer the question without a secant; the like will be also made evident in all the following cases.

4th. *Solution by Natural Sines.*

From the foregoing analogies, or statements, it is obvious that if the hypothenuse be multiplied by the natural sine of either of the acute angles, the product will be the length of the side opposite to that angle; and multiplied by the natural cosine of the same angle, the product will be the length of the other side, or that which is contiguous to the angle. Thus:

The given angle=47° 30′

Nat. Sine=.725374	Nat. Cos.=.688355
Hyp.= 250	250
36268700	34417750
1450748	1376710
Perpend.=181.343500	Base=172.088750

CASE II.

The base and angles given, to find the perpendicular and hypothenuse.

PL. 5. *fig.* 5.

In the triangle *ABC*, there is the angle *A* 42° 20′, and of course the angle *C* 47° 40′ (by cor. 2, theo. 5), and the side *AB* 190 given; to find *BC* and *AC*.

1st. *By Construction.*

Make the angle *CAB* (by prop. 16, sect. 4) in blank lines, as before. From a scale of equal parts lay 190 from *A* to *B*, on the point *B* erect a perpendicular *BC* (by prob. 5, sect. 4), the point where this cuts the other blank line of the angle will be *C*; so is the triangle *ABC* constructed: let *AC* and *BC* be measured from the same scale of equal parts that *AB* was taken from, and the answers are found.

2d. *By Calculation.*

1. *Making AC the radius.*

S.C : *AB* : : *R* : *AC*.
S.C : *AB* : : *S.A* : *BC*.

That is, as the sine of C	$=47°\ 40'$	9.868785
is to AB,	$=190$	2.278754
So is radius	$=90°$	10.000000
		12.278754
to AC,	$=257$	2.409969
As the sine of C	$=47°\ 40'$	9.868785
is to AB,	$=190$	2.278754
So is the sine of A	$=42°\ 20'$	9.828301
		12.107055
to BC,	$=173.1$	2.238270

2. *Making AB the radius.*

$R : AB :: T.A : BC.$

$R : AB :: \text{Sec. } A : AC.$

That is, as radius	$=90°$	10.000000
is to AB,	$=190$	2.278754
So is the tangent of A	$=42°\ 20'$	9.959516
to BC,	$=173.1$	2.238270
As radius	$=90°$	10.000000
is to AB,	$=190$	2.278754
So is the secant of A	$=42°\ 20'$	10.131215
to AC,	$=257$	2.409969

3. *Making BC the radius.*

$T.C : AB :: \text{Sec. } C : AC.$

$T.C : AB :: R : BC.$

That is, as the tangent of C	$=47°\ 10'$	10.040484
is to AB,	$=190$	2.278754
So is the secant of C	$=47°\ 40'$	10.171699
		12.450453
to AC,	$=257$	2.409969
As the tangent of C	$=47°\ 40'$	10.040484
is to AB,	$=190$	2.278754
So is the radius	$=90°$	10.000000
		12.278754
to BC,	$=173.1$	2.238270

Or, having found one of the required sides, the other may be obtained by one or the other of the cors. to theo. 14, sect. 4.

3d. *By Gunter's Scale.*

1. When AC is made the radius.

Extend from 47° 40′ to 90° on the line of sines; that distance will reach from 190 to 257, on the line of numbers, for AC.

2. When AB is made the radius, the first stating is thus performed:

Extend from 45° on the tangents (for the tangent of 45° is equal to the radius, or to the sine of 90° as before) to 42° 20′; that extent will reach from 190, on the line of numbers, to 173, for BC.

3. When BC is made the radius, the second stating is thus performed:

Extend from 47° 40′, on the line of tangents, to 45°, or radius; that extent will reach from 190 to 173, on the line of numbers, for BC; for the tangent of 47° 40′ is more than the radius, therefore the fourth number must be less than the second, as before.

The first two statings of this case answer the question without a secant.

4th. *Solution by Natural Sines.*

$$\frac{AB \times R}{S \text{ of } C.} = AC; \text{ and } \frac{AB \times S \text{ of } A}{S \text{ of } C} = BC.$$

Nat. $S.$ of $C.$ Side $AB \times R.$

Thus, .739239)190.000000(257.02, &c.$=AC.$

147 8478

4215220
3696195

5190250
5174673

1557700
1478478

and, .673443=Nat. S of $A.$
190=side $AB.$

60609870
673443

127.954170

Nat. *S* of *C*.739239)127.954170(173.09=*BC*.
73 9239

5403027
5174673

2283540
2217717

6502300
6653151

CASE III.

The angles and perpendicular given, to find the base and hypothenuse.

PL. 5. *fig.* 6.

In the triangle *ABC*, there is the angle *A* 40°, and consequently the angle *C* 50°, with *BC* 170, given, to find *AC* and *AB*.

1st. *By Construction.*

Make an angle *CAB* of 40° in blank lines (by prob. 16, sect. 4); with *BC* 170 from a line of equal parts draw the lines *EF* parallel to *AB* (by prob. 8, sect. 4), the lower line of the angle, and from the point where it cuts the other line in *C* let fall a perpendicular *BC* (by prob. 7, sect. 4), and the triangle is constructed: the measures of *AC* and *AB*, from the same scale that *BC* was taken, will answer the question.

What has been said in the two foregoing cases is sufficient to render the operations in this, both by calculation, Gunter's scale, and natural sines, so obvious, that it is needless to insert them; however, for the sake of the learner, we give for

Answers, *AC* 264.5, and *AB* 202.6.

CASE IV.

The base and hypothenuse given, to find the angles and perpendicular.

PL. 5. *fig.* 7.

In the triangle *ABC*, there is given *AB* 300 and *AC* 500; the angles *A* and *C* and the perpendicular *BC* are required.

1st. *By Construction.*

From a scale of equal parts lay 300 from A to B; on B erect an indefinite blank perpendicular line; with AC 500 from the same scale, and one foot of the compasses in A, cross the perpendicular line in C; and the triangle is constructed.

By prob. 17, sect. 4, measure the angle A, and let BC be measured from the same scale of equal parts that AC and AB were taken from; and the answers are obtained.

2d. *By Calculation.*

1. *Making AC the radius.*

$AC : R :: AB : S.C.$
$R : AC :: S.A : BC.$

That is, as AC	=500	2.698970
is to radius,	=90°	10.000000
So is AB	=300	2.477121
		12.477121
to the sine of C,	= 36° 52′	9.778151

By cor. 2, theo. 5, 90°—36° 52′=53° 08′, the angle A.

As radius	=90°	10.000000
is to AC,	=500	2.698970
So is the sine of A	=53° 08′	9.903108
to BC,	=400	2.602078

2. *Making AB the radius.*

$AB : R :: AC :: \text{sec. } A.$
$R : AB :: T.A : BC.$

That is, as AB	=300	2.477121
is to radius,	=90°	10.000000
So is AC	=500	2.698970
		12.698970
to the secant of A,	=53° 08′	10.221849
As radius	=90°	10.000000
is to AB,	=300	2.477121
So is the tangent of A	=53° 08′	10.124990
to BC,	=400	2.602111

Or BC may be found from cor. 2, theo. 14, sect 4.

3d. *By Gunter's Scale.*

1. *Making AC the radius.*

Extend from 500 to 300, on the line of numbers; that extent will reach from 90°, on the line of sines, to 36° 52′ for the angle *C*.

Again, extend from 90° to 53° 08′, on the line of sines, that extent will reach from 500 to 400, on the line of numbers, for *BC*.

2. *Making AC the radius*, the second stating is thus performed.

Extend from radius, or the tangent of 45°, to 53° 08′, that extent will reach from 300 to 400, for *BC*.

4th. *Solution by Natural Sines.**

$$\frac{R \times AB}{AC} = S \text{ of } C; \text{ and } \frac{AC \times S \text{ of } A}{R} = BC,$$

Thus, *AC*, *AB*,

5,00)300.0000,00

.600000=Nat. sine 36° 52′.

and,

Nat. sine of *A*=53° 8′=.800034

AC = 500

400.017000=*BC*.

CASE V.

The perpendicular and hypothenuse given, to find the angles and base.

Pl. 5. *fig.* 8.

In the triangle *ABC* there is *BC* 306 and *AC* 370 given, to find the angles *A* and *C* and the base *AB*.

1st. *By Construction.*

Draw a blank line from any point, in which at *B* erect a perpendicular, on which lay *BC* 306, from a scale of equal parts: from the same scale, with *AC* 370 in the compasses

* For finding the natural sines and co-sines, the reader is referred to table 3.

draw the first drawn blank line in A, and the triangle ABC is constructed.

Measure the angle A (by prob. 17, sect. 4), and also AB, from the same scale of equal parts the other sides were taken from, and the answers are now found.

The operations by calculation, the square root, Gunter's scale, and natural sines are here omitted, as they have been heretofore fully explained: the statings, or proportions, must also be obvious, from what has already been said.

Answers. The angle A 55° 48′; therefore the angle C 34° 12′, and AB 208.

CASE VI.

The base and perpendicular given, to find the angles and hypothenuse.

Pl. 5. *fig.* 9.

In the triangle ABC, there is AB 225 and BC 272 given, to find the angles A and C and the hypothenuse AC.

1st. *By Construction.*

Draw a blank line, on which lay AB 225, from a scale of equal parts; at B erect a perpendicular; on which lay BC 272 from the same scale; join A and C, and the triangle is constructed.

As before, let the angle A and the hypothenuse AC be measured, in order to find the answers.

2d. *By Calculation.*

1. *Making AB the radius.*

$$AB : R :: BC : T.A.$$
$$R : AB :: \text{sec. } A : AC.$$

2. *Making BC the radius.*

$$BC : R :: AB : T.C.$$
$$R : BC :: \text{sec. } C : AC.$$

By calculation, the answers from the foregoing proportions are easily obtained as before.

But because AC, by either of the said proportions, is found by means of a secant, and since there is no line of secants on Gunter's scale, after having found the angles as before, let us suppose AC the radius, and then

$$1.\ S.A : BC :: R : AC$$
$$\text{or } 2.\ S.C : AB :: R : AC.$$

These proportions may be easily resolved, either by calculation or Gunter's scale, as before; and thus the hypothenuse AC may be found without a secant.

From the two given sides the hypothenuse may be easily obtained, from cor. 1, theo. 14, sect.

Thus, the square of AB=50625
Add the square of BC=73984

124609(353=AC.
9

65)346
325

703)2109
2109

From what has been said on logarithms, it is plain,

1. That half the logarithm of the sum of the squares of the two sides will be the logarithm of the hypothenuse. Thus,*

The sum of squares, as before, is 124609; its log. is 5.095549, the half of which is 2.547774; and the corresponding number to this in the tables will be 353, for AC.

2. And that half of the logarithm of the difference of the squares of AC and AB, or of AC and BC, will be the logarithm of BC, or of AB.

The following examples are inserted for the exercise of the learner.

Ex. 1. In the right-angled triangle ABC,

Given, { the hypothenuse AC 540 perches, $\angle A$ 33° 45′ } Ans. { BC 300, AB 449 }

To find the other two sides.

Ex. 2. In the right-angled triangle ABC,

Given, { the base AB 162 chains, $\angle A$ 33° 45′ } Ans. { AC 270, BC 216 }

To find the other two sides.

* Demonstration. The square of the hypothenuse of a right-angled triangle is equal to the sum of the squares of the sides (theo. 14); hence the log. of $(AC^2 AB^2)$ = the log. of BC^2, and by the nature of logarithms the log. of BC is equal to the log. of BC^2 divided by 2; and in like manner the log. of (AB^2+BC^2) = the log. of AC^2, hence dividing the log. of AC^2 by 2 gives the log. of AC. Q. E. D.

Ex. 3. In the right-angled triangle ABC,

Given, { the perpendicular BC 180 links, $\angle C$ 62° 40′ } Ans. { AC 392.0146 AB 348.2464

To find the other two sides.

Ex. 4. In the right-angled triangle ABC,

Given, { the hypothenuse AC 392 poles, the base AB 180 poles } Ans. { $\angle A$ 62° 40′ $\angle C$ 27° 20′ BC 348.25

To find the angles and perpendicular.

Ex. 5. In the right-angled triangle ABC,

Given, { the hypothenuse AC 1198 chains, the perpendicular BC 980 chains } Ans. { $\angle A$ 54° 51′ $\angle C$ 35° 09′ AB 690

To find the angles and base.

Ex. 6. In the right-angled triangle ABC,

Given, { the base AB 735.9 links, the perpendicular BC 320 } Ans. { $\angle C$ 66° 30′ $\angle A$ 23° 30′ AC 802.5

To find the angles and hypothenuse.

OBLIQUE-ANGLED PLANE TRIGONOMETRY.

Before we proceed to the solution of the four cases of Oblique-angled triangles, it is necessary to premise the following theorems.

THEOREM I.

Pl. 5. *fig.* 10.

In any plane triangle ABC the sides are proportional to the sines of their opposite angles; that is, S.C : AB : : S.A : BC, and S.C : AB : : S.B : AC; also, S.B : AC : : S.A : BC.

By theo. 10, sect. 4, the half of each side is the sine of its opposite angle; but the sines of those angles, in tabular parts, are proportional to the sines of the same in any other measure; and therefore the sines of the angles will be as the halves of their opposite sides; and since the halves are as the wholes, it follows that the sines of their angles are as their opposite sides; that is, $S.C : AB : : S.A : BC$, &c. Q. E. D.

THEOREM II.

Fig. 11.

In any plane triangle ABC the sum of the two given sides AB and BC, including a given angle ABC, is to their difference as the tangent of half the sum of the two unknown angles A and C is to the tangent of half their difference.

Produce AB, and make $HB=BC$, and join HC: let fall the

perpendicular BE, and that will bisect the angle HBC (by theo. 9, sect. 4); through B draw BD parallel to AC, and make $HF=DC$, and join BF; take $BI=BA$, and draw IG parallel to BD or AC.

It is then plain that AH will be the sum and HI the difference of the sides AB and BC: and since $HB=BC$, and BE perpendicular to HC, therefore $HE=EC$ (by theo. 8, sect. 4); and since $BA=BI$, and BD and IG parallel to AC, therefore $GD=DC=FH$, and consequently $HG=FD$, and $\frac{1}{2}HG=\frac{1}{2}FD$ or ED. Again, EBC, being half HBC, will be also half the sum of the angles A and C (by theo. 4, sect. 4); also, since HB, HF, and the included angle H are severally equal to BC, CD, and the included angle BCD, therefore (by theo. 6, sect. 4) $HBF=DBC=BCA$ (by part 2, theo. 3, sect. 4); and since $HBD=A$ (by part 3, theo. 3, sect. 4), and $HBF=BCA$, therefore FBD is the difference and EBD half the difference of the angles A and C: then making BE the radius, it is plain that EC will be the tangent of half the sum, and ED the tangent of half the difference of the two unknown angles A and C: now IG being parallel to AC, $AH : IH :: CH : GH$ (by cor. 1, theo. 20, sect. 4). But the wholes are as their halves; that is, $AH : IH :: CE : ED$; that is, as the sum of the two sides AB and BC is to their difference, so is the tangent of half the sum of the two unknown angles A and C to the tangent of half their difference. Q. E. D.

THEOREM III.

Fig. 12.

In any right-lined plane triangle ABD, the base AD will be to the sum of the other sides AB, BD as the difference of those sides is to the difference of the segments of the base made by the perpendicular BE; viz. the difference between AE and ED.

Produce BD till $BG=AB$, the lesser leg; and on B as a centre, with the distance BG or BA, describe a circle $AGHF$, which will cut BD and AD in the points H and F; then it is plain that GD will be the sum, and HD the difference of the sides AB and BD; also, since $AE=EF$ (by theo. 8, sect. 4), therefore FD is the difference of AE, ED, the segments of the base; but (by theo. 17, sect. 4) $AD : GD :: HD : FD$; that is, the base is to the sum of the other sides as the difference of those sides is to the difference of the segments of the base. Q. E. D.

Cor. 1. In the above triangle the longest side is made the base, and then the perpendicular falls within the triangle; but if DF (the same construction remaining as in the above, only

joining BF) (fig. 3, plate 14), be considered the base of the triangle BDF, then BE is a perpendicular on the base produced; GD is equal to the sum of the sides BF, BD; HD is equal to their difference: also AD is equal to the sum of the segments DE, EF. But (by theo. 17, sect. 4) $FD \times AD = GD \times HD$, hence $FD : GD :: HD : AD$. That is, as the base is to the sum of the two sides, so is the difference of the sides to the sum of the segments of the base. Q. E. D.

Cor. 2. Hence (by calling any side the base) as the base is to the sum of the sides, so is the difference of the sides to the difference or sum of the segments of the base, according as the perpendicular falls within or without the triangle.*

THEOREM IV.

Fig. 13.

If to half the sum of two quantities be added half their difference, the sum will be the greatest of them; and if from half the sum be subtracted half their difference, the remainder will be the least of them.

Let the two quantities be represented by AB and BC (making one continued line), whereof AB is the greatest, and BC the least. Bisect the whole line AC in E, and make $AD = BC$; then it is plain that AC is the sum, and DB the difference of the two quantities, and AE or EC their half-sum, and DE or EB their half-difference. Now if to AE we add EB, we shall have AB the greatest quantity; and if from EC we take EB, we shall have BC the least quantity. Q. E. D.

Cor. Hence, if from the greatest of two quantities we take half the difference of them, the remainder will be half their sum; or if to half their difference be added the least quantity, their sum will be half the sum of the two quantities.

THEOREM V.

Pl. 14. *fig.* 4.

In any triangle the rectangle under two sides is to the rectangle under the semiperimeter, and its excess above the base, as the square of the radius to the square of the co-sine of half the contained angle.

In the triangle CBE, the perimeter being denoted by P, $CB \times CE : \frac{1}{2}P(\frac{1}{2}P - BE) :: R^2 : \text{cos. } \frac{1}{2}C^2$. Produce EC to A,

* The perpendicular falls within or without the triangle, according as the square of the greater side is less or greater than the sum of the squares of the less side and the base. For a demonstration of which the reader is referred to (Prop. 12, 13, B. 2) Simpson's Euclid.

making $CA=CB$; draw BD perpendicular to CE, bisect CE in H, and join AB.

Let CB be greater than EB, then (by theo. 3, fig. 12) $CE : CB+BE :: CB-BE : \frac{CB^2-BE^2}{CE}=2HD$, by adding half this to half the base $=CH$. The segment $CD=\frac{CB^2-BE^2+CE^2}{2.CE}$; to this adding CA, or CB, gives $AD=\frac{CB^2-BE^2+CE^2+2CE.CB}{2.CE}$ $=\frac{(CB+CE)^2-BE^2}{2CE}=\frac{CB+CE+BE\times CB+CE-BE}{2CE}$.

Again, $AD=AC+CD=CB+CD$; hence $AD^2=CB^2+2CB.CD+CD^2=2CB.AD$; also, $BD^2=CB^2-CD^2$; hence $AB^2=AD^2+BD^2=2.CB^2+2CB.CD=2CB\times(CB+CD)$ $=2CB.AD$; therefore, $AD.AB^2=2CB.AD^2$, or $\frac{P(\frac{1}{2}P-BE)}{CE}$. $AB^2=2CB.AD^2$, or $P(\frac{1}{2}P-BE).AB^2=CE.2CB.AD^2$, dividing both sides by 2; $CE.CB.AD^2=\frac{1}{2}P(\frac{1}{2}P-BE).AB^2$, consequently $CE.CB : \frac{1}{2}P(\frac{1}{2}P-BE) :: AB^2 : AD^2$. That is, $CE\times CB : \frac{1}{2}P\times\frac{1}{2}(P-BE) :: \text{rad.}^2 : (\text{cos. } \frac{1}{2}BCE)^2$.* Q. E. D.

OBLIQUE-ANGLED TRIANGLES.

CASE I.

Two sides and an angle opposite to one of them given, to find the other angles and side.

PL. 5. *fig.* 14.

In the triangle ABC, there is given AB 240, *the angle A* 46° 30′, *and BC* 200, *to find the angle C, being acute, the angle B, and the side AC.*

1st. *By Construction.*

Draw a blank line, on which set AB 240, from a scale of equal parts; at the point A, of the line AB, make an angle of 46° 30′, by an indefinite blank line; with BC 200, from a like scale of equal parts that AB was taken, and one foot in B, describe the arc DC to cut the last blank line in the points D and C. Now if the angle C had been required obtuse, lines from D to B, and to A, would constitute the triangle; but as it is required acute, draw the lines from C to B and to A, and the triangle ABC is constructed. From a line of chords let the angles B and C be measured; and AC from the same scale

* For a different method of demonstrating this theorem, as well as the demonstration of other useful theorems, the reader is referred to Leslie's Geometry (pages 372, 373).

of equal parts that AB and BC were taken; and you will have the answers required.

2. *By Calculation.*

This is performed by theo. 1 of this sect. thus:

As BC	$=200$	2.301030
is to the sine of A,	$=46°30'$	9.860562
So is AB	$=240$	2.380211
		12.240773
to the sine of C,	$=60°\ 31'$	9.939743

180° — the sum of the angles A and C will give the angle B, by cor. 1, theo. 5, sect. 4.

A 46° 30′
C 60 31

180°—107° 1′=72° 59′=B.

As the sine of A	$=46°\ 30'$	9.860562
is to BC,	$=200$	2.301030
So is the sine of B	$=72°\ 59'$	9.980555
		12.281585
to AC,	$=263.\ 7$	2.421023

3d. *By Gunter's Scale.*

Extend from 200 to 240 on the line of numbers; that distance will reach from 46° 30′, on the line of sines, to 60° 31′, for the angle C.

Extend from 46° 30′ to 72° 59′, on the line of sines; that distance will reach from 200 to 263.7 on the line of numbers, for AC.

Note.—The method by natural sines will be obvious from the foregoing analogies.

If the side opposite the given angle be equal to or greater than the other given side, or the given angle obtuse, then there would be but one answer to the problem, because the angle opposite that other given side will be always acute; but when the given angle is acute, and opposite the less of the given sides, the answer is ambiguous, as the sine of an angle is equal to the sine of its supplement, consequently the required angle opposite that other given side may be obtuse or acute, unless it is given in the conditions of the problem.

In the last problem the given angle is acute, and the side opposite to it less than the other given side, therefore the angle C may be acute or obtuse; but the side and angle answering to the acute value of C has been already found. Now it remains to find the side and angle of the triangle answering to the obtuse value of C, which is thus found:

The acute value of C, found in the foregoing calculation, is 60° 31′, consequently its obtuse value is 180°—60° 31′=119° 29′; then 119° 29′+46° 30′, taken from 180°, gives 14° 1′= to the remaining angle ABC (pl. 14, fig. 5).

To find the side AC, answering to the obtuse value of the angle C.

As the sine of A	=46° 30′	9.860562
is to BC,	=200	2.301030
So is sine B	=14° 1′	9.384182
		11.685212
to the side AC,	=66.8	1.824650

CASE II.

Two angles and a side given, to find the other sides.

PL. 5. *fig.* 15.

In the triangle ABC there is the angle A 46° 30′, *AB* 230, *and the angle B* 37° 30′ *given, to find AC and BC.*

1st. *By Construction.*

Draw a blank line, upon which set AB 230 from a scale of equal parts; at the point A of the line AB make an angle of 46° 30′, by a blank line; and at the point B of the line AB make an angle of 37° 30′, by another blank line; the intersection of those lines gives the point C: then the triangle ABC is constructed. Measure AC and BC from the same scale of equal parts that AB was taken, and you have the answer required.

2d. *By Calculation.*

By cor. 1, theo. 5, sect 4, 180° — the sum of the angles A and B=C.

A	46° 30′
B	37 30
180°—84° 00′=96° 00′=C.	

By def. 27, sect. 4. The sine of 96° = the sine of 84°, which is the supplement thereof; therefore, instead of the sine of 96°, look in the tables for the sine of 84°.

By theo. 1 of this sect.

As the sine of C	=96° 00′	9.997614
is to AB,	=230	2.361728
So is the sine of A	=46° 30′	9.860562
		12.222290
to BC,	=167.8	2.224676
As the sine of C	=96° 00′	9.997614
is to AB,	=230	2.361728
So is the sine of A	=37° 30′	9.784447
		12.146175
to AC,	=140.8	2.148561

3d. *By Gunter's Scale.*

Extend from 84° (which is the supplement of 96°) to 46° 30′ on the sines; that distance will reach from 230 to 168, on the line of numbers, for BC.

Extend from 84° to 37° 30′, on the sines; that extent will reach from 230 to 141, on the line of numbers, for AC.

CASE III.

Two sides and a contained angle given, to find the other angles and side.

PL. 5. *fig.* 16.

In the triangle ABC, there is AB 240, *the angle A* 36° 40′, *and AC* 180 *given, to find the angles C and B and the side BC.*

1st. *By Construction.*

Draw a blank line, on which, from a scale of equal parts, lay AB 240; at the point A of the line AB make an angle of 36° 40′, by a blank line; on which from A lay AC 180, from the same scale of equal parts; measure the angles C and B and the side BC, as before, and you have the answers required.

2d. *By Calculation.*

By cor. 1, theo. 5, sect. 4, 180° — the angle A 36° 40′= 143° 20′, the sum of the angles C and B: therefore, half of

143° 20′ will be half the sum of the two required angles *C* and *B*.

By theo. 2 of this sect.

As the sum of the two sides *AB* and *AC* = 420
is to their difference, = 60
So is the tangent of half the sum of the two unknown angles *C* and *B* = 71° 40′
to the tangent of half their difference, = 23° 20′

By theo. 4.

To half the sum of the angles *C* and *B* = 71° 40′
Add half their difference as now found = 23° 20′

The sum is the greatest angle, or ang. *C* = 95 00

Subtract, and you have the least angle, or *B* = 48 20

The angles *C* and *B* being found, *BC* is had as before, by theo. 1 of this sect. Thus,

S.B : *AC* : : *S.A* : *BC*.
48° 20′ : 180 : : 36° 40′ : 143.9.

3d. *By Gunter's Scale.*

Because the two first terms are of the same kind, extend from 420 to 60 on the line of numbers; lay that extent from 45° on the line of tangents, and keeping the left leg of your compasses fixed, move the right leg to 71° 40′; that distance laid from 45° on the same line will reach to 23° 30′, the half-difference of the required angles. Whence the angles are obtained, as before.

The second proportion may be easily extended, from what has been already said.

CASE IV.

PL. 5. *fig.* 17.

The three sides given, to find the angles.

In the triangle ABC, there is given AB 64, *AC* 47, *BC* 34; *the angles A, B, and C are required.*

1st. *By Construction.*

The construction of this triangle must be manifest, from prob. 1, sect. 4.

2d. *By Calculation.*

From the point *C* let fall the perpendicular *CD* on the base *AB*, and it will divide the triangle into two right-angled ones, *ADC* and *CBD*, as well as the base *AB* into the two segments *AD* and *DB*.

AC	47
BC	34
Sum	81
Difference	13

By theo. 3 of this sect.

As the base or the longest side *AB*	64
is to the sum of the other sides *AC* and *BC*,	81
So is the difference of those sides	13
to the difference of the segments of the base *AD*, *DB*,	16.46

By theo. 4 of this sect.

To half the base, or to half the sum of the segments *AD* and *DB*,	32
Add half their difference, now found,	8.23
Their sum will be the greatest segment *AD*,	40.23
Subtract, and their difference will be the least segment, *DB*,	23.77

In the right-angled triangle *ADC*, there is *AC* 47 and *AD* 40.23 given, to find the angle *A*.

This is resolved by case 4 of right-angled plane trigonometry, thus:

AD : *R* : : *AC* : sec. *A*
40.23 : 90° : : 47 : 31° 08′

Or it may be had by finding the angle *ACD*, the complement of the angle *A*, without a secant, thus:

AC : *R* : : *AD* : *S.ACD*
44 : 90° : : 40.23 : 58° 52′
90°—58° 52′=31° 08′, the angle *A*.

Then by theo. 1 of this sect.

BC : *S.A* : : *AC* : *S.B*
34 : 31° 08′ : : 47 : 45° 37′

By cor. 1, theo. 5, sect. 4, 180° — the sum of A and $B = C$.

A 31° 08′
B 45 37

180°—76 45=103° 15′, the angle C.

3d. *By Gunter's Scale.*

The first proportion is extended on the line of numbers; and it is no matter whether you extend from the first to the third or to the second term, since they are all of the same kind: if you extend to the second, that distance applied to the third will give the fourth; but if you extend from the first to the third, that extent will reach from the second to the fourth.*

The methods of extending the other proportions have been already fully treated of.

RULE 2.

Either of the angles, as A, may be found by adding together the arithmetical complements of the logarithms of the two sides AB, AC, containing the required angle, the log. of the half-sum of the three sides, and the log. of the difference between the half-sum and the side opposite the required angle; then half the sum of these four logarithms will be the logarithmic co-sine of half the required angle.† It is required to find the angle A, in the last problem, by this rule, the sides remaining the same.

BC=34	
AC=47	Ar. Co. 7.327902
AB=64	Ar. Co. 7.193820
2)145	
Half-sum 72.5	Log. 2.860338
BC=34	
Difference 38.5	Log. 2.585461
	2)19.967521
Cos. ½ A, 15° 34′	9.983760

Whose double 31° 08′ is the angle A.

* The reader is referred to Hutton's Mathematics, vol. ii. New-York edition, for the method of investigating Plane Trigonometry analytically.

† The demonstration of this rule is evident from theo. 5, and the nature of logarithms; but in working the proportion by logarithms, we omit

If the other angles were required, they can be found by Case 1, or by theo. 1 of this sect.

RULE 3.*

Add the three sides together, and take half the sum and the differences between the half-sum and each side: then add the complements of the logarithms of the half-sum and of the difference between the half-sum and the side opposite to the angle sought, to the logarithms of the differences of the half-sum and the other sides: half their sum will be the tangent of the angle required.

Example. In the triangle ABC, having the side AB 562, AC 800, and BC 320, to find the angle ABC.

AC=800	H=841 Ar. Co.	7.075204
AB=562	$H-AC$=41 Ar. Co.	8.387216
BC=320	$H-AB$=279 log.	2.445604
Sum 1682	$H-BC$=521 log.	2.716838
		Sum 20.624862

$\frac{1}{2}$ sum 841=H.

Tang. of 64° 2′=$\frac{1}{2}$ sum 10.312431

Whose double 128° 4′ is the angle ABC. Whence the other angles can be easily found by theo. 1 of this section.

An example in each case of oblique-angled triangles.

1. In the triangle ABC, having AB 106, AC 65, and the angle B 31° 49′, to find the ∠s A and C and the side BC.

Ans. The ∠ C=59° 17′ or 120° 43′, the ∠A 27° 28′ or 88° 54′, and the side BC=43.2 or 123.2.

2. In the triangle ABC, having the side AB 2200, the ∠A 35°, and the ∠B 47° 24′, to find the sides AC and BC and the ∠C.

Ans. The ∠C 97° 36′, the side AC 1636, and the side BC 1272.

3. In the triangle ABC, having the side AB 240, AC

---

the log. of the square of radius or 20, which is just equivalent to rejecting 20 from the sum of the four logarithms, which should be done, because for every arithmetical complement that is taken 10 must be rejected: but the Ar. Co. of the two sides containing the required angle is taken; consequently 20 should be rejected, which is equal to the log. of the square of radius.

* For the demonstration of this rule the reader is referred to Leslie's Geometry, prop. 12, p. 372.

263.7, and the angle A 46° 30′, to find the other angles and the side BC.

Ans. The $\angle C$ 60° 31′, the $\angle B$ 72° 59′, and the side BC 200.

4. In the triangle ABC, having the sides given, viz. AB= 144.8, BC=109, and AC=76, it is required to find the angles by each of the three rules given to Case 4.

Ans. The least angle 29° 49′, next greater 54° 07′, and the greatest 96° 04′.

Additional exercises, with their answers.

QUESTIONS FOR EXERCISE.

1. Given the hypothenuse 108, and the angle opposite the perpendicular 25° 36′; required the base and perpendicular.

Ans. The base is 97.4, and the perpendicular 46.66.

2. Given the base 96, and its opposite angle 71° 45′; required the perpendicular and the hypothenuse.

Ans. The perpendicular is 31.66, and the hypothenuse 101.1.

3. Given the perpendicular 360, and its opposite angle 58° 20′; required the base and the hypothenuse.

Ans. The base is 222, and the hypothenuse 423.

4. Given the base 720, and the hypothenuse 980; required the angles and the perpendicular.

Ans. The angles are 47° 17′ and 42° 43′, and the perpendicular 664.8.

5. Given the perpendicular 110.3, and the hypothenuse 176.5; required the angles and the base.

Ans. The angles are 38° 41′ and 51° 19′, and the base 137.8.

6. Given the base 360, and the perpendicular 480; required the angles and the hypothenuse.

Ans. The angles are 53° 8′ and 36° 52′, and the hypothenuse 600.

7. Given one side 129, an adjacent angle 56° 30′, and the opposite angle 81° 36′; required the third angle and the remaining sides.

Ans. The third angle is 41° 54′, and the remaining sides are 108.7 and 87.08.

8. Given one side 96.5, another side 59.7, and the angle opposite the latter side 31° 30′; required the remaining angles and the third side.

Ans. This question is ambiguous, the given side opposite the given angle being less than the other given side (see Rule 1);

hence, if the angle opposite the side 96.5 be acute, it will be 57° 38′, the remaining angle 90° 52′, and the third side 114.2; but if the angle opposite the side 96.5 be obtuse, it will be 122° 22′, the remaining angle 26° 8′, and the third side 50.32.

9. Given one side 110, another side 102, and the contained angle 113° 36′; required the remaining angles and the third side.

Ans. The remaining angles are 34° 37′ and 31° 47′, and the third side is 177.5.

10. Given the three sides respectively 120.6, 125.5, and 146.7; required the angles.

Ans. The angles are 51° 53′, 54° 58′, and 73° 9′.

The student who has advanced thus far in this work with diligence and active curiosity is now prepared to study, with ease and pleasure, the following Part, which comprehends all the necessary directions for the practice of Surveying.

PART II.

THE PRACTICAL SURVEYOR'S GUIDE.

SECTION I.

Containing a particular Description of the several Instruments used in Surveying, with their respective Uses.

THE CHAIN.

The stationary distance, or merings of ground, are measured either by Gunter's chain of four poles or perches, which consists of 100 links (and this is the most natural division), or by one of 50 links, which contains two poles or perches: but because the length of a perch differs in many places, therefore the length of chains and their respective links will differ also.

The *English statute-perch* is 5½ yards, the two-pole chain is 11 yards, and the four-pole one is 22 yards; hence the length of a link in a statute-chain is 7.92 inches.

For the more ready reckoning the links of a four-pole chain, there is a large ring, or sometimes a round piece of brass, fixed at every 10 links; and at 50 links, or in the middle, there are

two large rings. In such chains as have a brass piece at every 10 links, there is the figure 1 on the first piece, 2 on the second, 3 on the third, &c. to 9. By leading therefore that end of the chain forward which has the least number next to it, he who carries the hinder end may easily determine any number of links: thus, if he has the brass piece number 8 next to him, and six links more in a distance, that distance is 86 links. After the same manner 10 may be counted for every large ring of a chain which has not brass pieces on it; and the number of links is thus readily determined.

The two-pole chain has a large ring at every 10 links, and in its middle, or at 25 links, there are two large rings; so that any number of links may be the more readily counted off, as before.

The surveyor should be careful to have his chain measured before he proceeds on business; for the rings are apt to open by frequently using it, and its length is thereby increased, so that no one can be too circumspect in this point.

In measuring a stationary distance, there is an object fixed in the extreme point of the line to be measured; this is a direction for the hinder chainman to govern the foremost one by, in order that the distance may be measured in a right line; for if the hinder chainman causes the other to cover the object, it is plain the foremost is then in a right line towards it. For this reason it is necessary to have a person that can be relied on at the hinder end of the chain, in order to keep the foremost man in a right line; and a surveyor who has no such person should chain himself. The inaccuracies of most surveys arise from bad chaining, that is, from straying out of the right line, as well as from other omissions of the hinder chainman: no person, therefore, should be admitted at the hinder end of the chain of whose abilities, in this respect, the surveyor is not previously convinced; since the success of the survey, in a great measure, depends on his care and skill.

In setting out to measure any stationary distance, the fore man of the chain carries with him ten iron pegs pointed, each about ten inches long; and when he has stretched the chain to its full length, he at the extremity thereof sticks one of those pegs perpendicularly in the ground; and leaving it there, he draws on the chain till the hinder man checks him when he arrives at that peg: the chain being again stretched, the fore man sticks down another peg, and the hind man takes up the former; and thus they proceed at every chain's length contained in the line to be measured, counting the surplus links contained between the last peg and the object at the termination of the line, as before: so that the number of pegs taken

up by the hinder chainman expresses the number of chains: to which, if the odd links be annexed, the distance line required in chains and links is obtained, which must be registered in the field-book, as will hereafter be shown.

If the distance exceeds 10, 20, 30, &c. chains, when the leader's pegs are all exhausted, the hinder chainman, at the extremity of the 10 chains, delivers him all the pegs; from whence they proceed to measure as before, till the leader's pegs are again exhausted, and the hinder chainman at the extremity of these 10 chains again delivers him the pegs, from whence they proceed to measure the whole distance line in the like manner; then it is plain, that the number of pegs the hinder chainman has being added to 10, if he had delivered all the pegs once to the leader, or to 20 if twice, or to 30 if thrice, &c., will give the number of chains in that distance; to which if the surplus links be added, the length of the stationary distance is known in chains and links.

It is customary, and indeed necessary, to have red, or other coloured cloth fixed to the top of each peg, that the hinder man at the chain may the more readily find them; otherwise, in chaining through corn, high grass, briers, rushes, &c. it would be extremely difficult to find the pegs which the leader puts down: by this means no time is lost, which otherwise must be, if no cloths are fixed to the pegs, as before.

It will be necessary here to observe, that all slant, or inclined surfaces, as sides of hills, are measured horizontally, and not on the plane or surface of the hill, and is thus effected.

PL. 8. *fig.* 4.

Let *ABC* be a hill; the hindmost chainman is to hold the end of the chain perpendicularly over the point *A* (which he can the better effect with a plummet and line, than by letting a stone drop, which is most usual), as *d* is over *A*, while the leader puts down his peg at *e*: the eye can direct the horizontal position near enough; but if greater accuracy were required, a quadrant applied to the chain would settle that. In the same manner the rest may be chained up and down; but in going down, it is plain the leader of the chain must hold up the end thereof, and the plummet thence suspended will mark the point where he is to stick his peg. The figure is sufficient to render the whole evident, and to show that the sum of the chains will be the horizontal measure of the base of the hill: for $de = Ao$, $fg = op$, $hi = pq$, &c.; therefore $de + fg + hi$, &c. $= Ao + op + pq$, &c.

$=AC$,* the base of the hill. If a whole chain cannot be carried horizontally, half a chain, or less, may, and the sum of these half-chains, or links, will give the base, as before.

If the inclined side of the hill be the plane surface, the angle of the hill's inclination may be taken, and the slant height may be measured on the surface; and thence (by Case 1 of right-angled trigonometry) the horizontal line answering to the top may be found; and if we have the angle of inclination given on the other side, with those already given, we can find the horizontal distance across the hill, by Case 2 of oblique trigonometry.

All inclined surfaces are considered as horizontal ones; for all trees which grow upon any inclined surface do not grow perpendicular thereto, but to the plane of the horizon: thus, if *Ad*, *ef*, *gh*, &c. were trees on the side of a hill, they grow perpendicular to the horizontal base AC, and not to the surface AB: hence the base will be capable to contain as many trees as are on the surface of the hill, which is manifest from the continuation of them thereto. And this is the reason that the area of the base of a hill is considered to be equal in value to the hill itself.

Besides, the irregularities of the surfaces of hills in general are such, that they would be found impossible to be determined by the most able mathematicians. Certain regular curve surfaces have been investigated, with no small pains, by the most

* The number of chains taken down in the field-book is longer than the lines *Ao*, *op*, *pq*, &c., because the chain, being elevated above the surface of the earth (though stretched with a force at both ends), forms a curve, which approaches a right line, according as the force is more or less applied; but does not coincide with it: as, for example.—Let the chain be stretched from *d* to *e* (Pl. 8. fig. 4); it does not coincide with *de*, but forms a curve line, which must be longer than *de* or its equal *Ao*, and so is *fg* or *op* shorter than the chain, and in like manner with all the rest. And *de*, *fg*, &c.$=Ao$, *op*, &c.$=AC$; consequently, the number of chains, being greater than *de*, *ef*, &c. or *Ao*, *op*, &c. is greater than AC; therefore, the horizontal line AC (by surveyors in general) is made too long, therefore a deduction must be made for every chain in the field-book; the sum to be taken from AC may be found by making an experiment on a two-pole chain (when extended above the surface of the earth by a force at both extremities), and measuring the distance from its middle point to the middle of the right line which would join its extremities, which call a, and call $\frac{1}{2}$ the length of the chain b; then $\sqrt{b^2-a^2}=\frac{1}{2}de$, or half the right line; therefore $2\sqrt{b^2-a^2}=de$, or the right line; from whence $2b-2\sqrt{b^2-a^2}=$ the excess for every chain which is measured or taken down in the field-book: calling the number of chains c, then $c.2b-2\sqrt{(b^2-a^2)}=$ the whole excess on the horizontal line AC. From what is here demonstrated, the practitioner will be able to find the sum to be taken from every horizontal line in surveying hills, &c.

eminent; therefore an attempt to determine in general the infinity of irregular surfaces which offer themselves to our view, to any degree of certainty, would be idle and ridiculous, and for this reason also, the horizontal area is only attempted.

Again, if the circumjacent lands of a hill be planned or mapped, it is evident we shall have a plan of the hill's base in the middle: but were it possible to put the hill's surface in lieu thereof, it would extend itself into the circumjacent lands, and render the whole a heap of confusion: so that if the surfaces of hills could be determined, no more than the base could be mapped.

Roads are usually measured by a wheel for that purpose, called the perambulator, to which there is fixed a machine, at the end whereof there is a spring, which is struck by a peg in the wheel once in every rotation; by this means the number of rotations is known; if such a wheel were 3 feet 4 inches diameter, one rotation would be 10¼ feet, which is half a plantation perch; and because 320 perches make a mile, therefore 640 rotations will be a mile also; and the machinery is so contrived that by means of a hand, which is carried round by the work, it points out the miles, quarters, and perches, or sometimes the miles, furlongs, and perches.

Or roads may be measured by a chain more accurately; for 80 four-pole, or 160 two-pole chains, or 320 perches, make a mile as before: and if roads are measured by a statute chain, it will give you the miles English, but if by a plantation chain, the miles will be Irish. Hence an English mile contains 1760, and an Irish mile 2240 yards; and because 14 half-yards is an Irish, and 11 half-yards is an English perch, therefore 11 Irish perches, or Irish miles, are equal to 14 English ones.

Since some surveys are taken by a four-pole and others by a two-pole chain, and as ground for houses is measured by feet, we will show how to reduce one to the other in the following problems.

PROBLEM I.

To reduce two-pole chains and links to four-pole ones.

If the number of chains be even, the half of them will be the four-pole ones, to which annex the given links. Thus:

1. In 16*ch.* 37*l.* of two-pole chains, how many four-pole ones? Answer 8*ch.* 37*l.*

But if the number of chains be odd, take the half of them for

chains, and add 50 to the links, and they will be four-pole chains and links. Thus:

2. In 17*ch.* 42*l.* of two-pole chains how many four-pole ones? Answer 8*ch.* 92*l.*

PROBLEM II.

To reduce four-pole chains and links to two-pole ones.

Double the chains, to which annex the links if they be less than 50; but if they exceed 50 double the chains, add one to them, and take 50 from the links, and the remainder will be the links. Thus:

ch. l.

1. In 8. 37 of four-pole chains how many two-pole ones?

2

16. 37

ch. l.

2. In 8. 82 of four-pole chains how many two-pole ones?

2. 50

17. 32, Answer.

PROBLEM III.

To reduce four-pole chains and links to perches and decimals of a perch.

The links of a four-pole chain are decimal parts of it, each link being the hundredth part of a chain; therefore if the chain and links be multiplied by 4 (for 4 perches are a chain), the product will be the perches and decimal parts of a perch. Thus:

ch. l.

How many perches in 13. 64 of four-pole chains?

4

Answer, 54. 56 perches.

PROBLEM IV.

To reduce two-pole chains and links to perches and decimals of a perch.

They may be reduced to four-pole ones (by prob. 1), and thence to perches and decimals (by the last); or,

If the links be multiplied by 4, carrying one to the chains when the links are, or exceed, 25; and the chains by 2, adding 1 if

occasion be; the product will be the perches and decimals of a perch. Thus:

ch. l.

1. In 17. 21 of two-pole chains how many perches?
2. 4

Answer, 34. 84 perches.

ch. l.

2. In 15. 38 of two-pole chains how many perches?
2. 4

Answer, 31. 52 perches.

PROBLEM V.

To reduce perches and decimals of a perch to four-pole chains and links.

Divide by 4, so as to have two decimal places in the quotient, and that will be four-pole chains and links. Thus:

In 31.52 perches how many four-pole chains and links?

ch. l.

4)31.52(7. 88, Answer.

35

32

PROBLEM VI.

To reduce perches and decimals of a perch to two-pole chains and links.

The perches may be reduced to four-pole chains (by the last), and from thence to two-pole chains (by prob. 2); or,

Divide the whole number by 2, the quotient will be chains; to the remainder annex the given decimals, and divide by 4; the last quotient will be the links. Thus:

In 31.52 perches how many two-pole chains and links?

ch. l.

2)31.52(15. 38, Answer.

11

4)152(38

32

PROBLEM VII.

To reduce chains and links to feet and decimal parts of a foot.

If they be two-pole chains, reduce them to four-pole ones (by prob. 1): these being multiplied by the feet in a four-pole chain will give the feet and decimals of a foot. Thus:

In 17*ch.* 21*l.* of two-pole chains how many feet?

ch. l.
8. 71 of four-pole chains.
66 feet = 1 chain.

5226
5226 Answer, 574*ft.* 10¼*in.*

Feet 574.86
12

Inches 10.32
4

1.28

PROBLEM VIII.

To reduce feet and inches to chains and links.

Reduce the inches to the decimal of a foot, and annex that to the feet; that divided by the feet in a four-pole chain will give the four-pole chains and links in the quotient; these may be reduced to two-pole chains and links, if required, by prob. 2. Thus:

In 217*ft.* 9*in.* how many two-pole chains?

12)9.00(.75 the decimal of 9 inches.
60

66)217.75(3.29 of four-pole chains, or 6*ch.* 29*l.*
197
655
61

How to take a survey by the CHAIN *only.*

PROBLEM I.

To survey a piece of ground by going round it, and the method of taking the angles of the field by the chain only.

PL. 6. *fig.* 6.

Let *ABCDEFG* be a piece of ground to be surveyed: beginning at the point *A*, let one chain be laid in a direct line from *A* towards *G*, where let a peg be left, as at *c*; and again the like distance from *A* in a direct line towards *B*, where another peg is also to be left, as at *d*; let the distance from *d* to *c* be measured, and placed in the field-book in the second column under the denomination of angles, in a line with station No. 1; and in the same line, under the title of distances in the third column, let the measure of the line *AB* in chains and links be inserted. Being now arrived at *B*, let one chain be laid in a direct line from *B* towards *A*, where let a peg be left, as at *f*, and again the like distance from *B* in a direct line towards *C*, where let also another peg be left, as at *e*; the distance from *e* to *f* is to be inserted in the field-book, in the second column, under angles, in a line with station No. 2; and in the same line, under the title of distances in the third column, let the measure of the line *BC*, in chains and links, be inserted: after the same manner we may proceed from *C* to *D*, and thence to *E*; but because the angle at *E*, viz. *FED*, is an external angle, after having laid one chain from *E* to *h*, and to *g*, the distance from *g* to *h* is measured and inserted in the column of angles, in a line with station No. 5, and on the side of the field-book against that station we make an asterisk, thus *, or any other mark, to signify that to be an external angle, or one measured out of the ground. Proceed we then as before from *E* to *F*, to *G*, and thence to *A*, measuring the angles and distances, and placing them as before in the field-book opposite to their respective stations: so will the field-book be completed in the manner following.

N. B.—After this manner the angles for inaccessible distances may be taken, and the method of constructing or laying them down, as well as the construction of the map, from the following field-notes, must be obvious from the method of taking them.

The form of the field-book, with the title.

A Field-Book of part of the land of Grange, in the parish of Portmarnock, barony of Coolock, and county of Dublin; being part of the estate of L. P., Esq., let to C. D., farmer. Surveyed January 30, 1782.

Taken by a four-pole chain.

Remarks.	No. Sta.	Angles. ch. l.	Distance. ch. l.
Mr. J. D.'s part of Grange . . .	1	1. 80	17. 65
	2	1. 79	18. 50
Mr. L. P.'s part of Portmarnock	3	1. 76	28. 00
Strand	4	1. 41½	20. 00
	5	1. 87½	14. 83
Widow J. G.'s part of Grange .	6	1. 14	19. 41
	7	1. 89	24. 53

Close at the first station.

Explanation of the Remarks.

Mr. J. D's part of Grange bounds or is adjacent to the surveyed land from the first to the third station; Mr. L. P's part of Portmarnock bounds it from the third to the fourth station; the strand then is the boundary from thence to the sixth; and from the sixth to the first station, the widow J. G's part of Grange is the boundary.

It is absolutely necessary to insert the persons' names, and town-lands, strands, rivers, bogs, rivulets, &c. which bound or circumscribe the land which is surveyed, for these must be expressed in the map.

In a survey of a town-land, or estate, it is sufficient to mention only the circumjacent town-lands, without the occupiers' names: but when a part only of a town-land is surveyed, then it is necessary to insert the person or persons' names who hold any particular parcel or parcels of such town-land as bound the part surveyed.

When an angle is very obtuse, as most in our present figure are, viz. the angles at *A*, *B*, *C*, *E*, and *G*, it will be best to lay a chain from the angular point, as at *A*, on each of the containing sides to *c* and to *d*; and any where nearly in the middle of the angle, as at *e*: measuring the distances *ce* and *ed*; and these may be placed for the angle in the field-book. Thus,

No. Sta.	Angle. ch. l.	ch. l.
	1. 03	17. 65
	1. 09	

For when an angle is very obtuse, the chord line, as *cd*, will be nearly equal to the radii *Ac* and *Ad*; so if the arc *ced* be swept, and the chord line *cd* be laid on it, it will be difficult to deter-

mine exactly that point in the arc where *cd* cuts it: but if the angle be taken in two parts, as *ce* and *ed*, the arc, and the angle thence, may be truly determined and constructed.

After the same manner any piece of ground may be surveyed by a two-pole chain.

PROBLEM II.

To take a survey of a piece of ground from any point within it, from whence all the angles can be seen, by the chain only.

Pl. 6. *fig.* 6.

Let a mark be fixed at any point in the ground, as at *H*, from whence all the angles can be seen; let the measures of the lines *HA*, *HB*, *HC*, &c. be taken to every angle of the field from the point *H*; and let those be placed opposite to No. 1, 2, 3, 4, &c. in the second column of the radii: the measures of the respective lines of the mering, viz. *AB*, *BC*, *CD*, *DE*, &c. being placed in the third column of distances, will complete the field-book. Thus:

Remarks.	No.	Radii. *ch.* *l.*	Distance. *ch.* *l.*
	1	20. 00	17. 65
	2	21. 72	18. 50
	3	21. 74	28. 00
	4	25. 34	20. 00
	5	17. 20	14. 83
	6	29. 62	19. 41
	7	21. 20	24. 53
		Close at the first station.	

If any line of the field be inaccessible, as suppose *CD* to be, then by way of proof that the distance *CD* is true, let the measure of the angle *CHD* be taken by the line *oo*, with the chain: if this angle corresponds with its containing sides, the length of the line *DC* is truly obtained, and the whole work is truly taken.

Note.—That in setting off an angle, it is necessary to use the largest scale of equal parts, viz. that of the inch, which is diagonally divided into 100 parts, in order that the angle should be accurately laid down; or if two inches were thus divided for angles, it would be the more exact; for it is by no means necessary that the angles should be laid from the said scale with the stationary distances.

PROBLEM III.

To take a survey by the chain only, when all the angles cannot be seen from one point within.

PL. 6. *fig.* 7.

Let the ground to be surveyed be represented by 1, 2, 3, 4, &c. Since all the angles cannot be seen from one point, let us assume three points, as *A*, *B*, *C*, from whence they may be seen; at each of which let a mark be put, and the respective sides of the triangle be measured and set down in the field-book; let the distance from *A* to 1, and from *B* to 1, be measured, and these will determine the point 1; let the other lines which flow from *A*, *B*, *C*, as well as the circuit of the ground, be then measured as the figure directs; and thence the map may be easily constructed.

There are other methods which may be used; as dividing the ground into triangles, and measuring the three sides of each; or by measuring the base and perpendicular of each triangle. But this we shall speak of hereafter.

PROBLEM IV.

How to take any inaccessible distance by the chain only.

PL. 8. *fig.* 8.

Suppose *AB* to be the breadth of a river, or any other inaccessible distance, which may be required.

Let a staff or any other object be set at *B*, draw yourself backward to any convenient distance *C*, so that *B* may cover *A*; from *B*, lay off any other distance by the river's side to *E*, and complete the parallelogram *EBCD*: stand at *D*, and cause a mark to be set at *F*, in the direction of *A*; measure the distance in links from *E* to *F*, and *FB* will be also given. Wherefore *EF* : *ED* : : *FB* : *AB*. Since it is plain (from part 1, theo. 3, sect. 4, and theo. 2, sect. 4) the triangles *EFD* and *BFA* are mutually equiangular.

If part of the chain be drawn from *B* to *C*, and the other part from *B* to *E*; and if the ends at *E* and *C* be kept fast, it will be easy to turn the chain over to *D*, so as to complete a parallelogram; by reckoning off the same number of links you had in *BC*, from *E* to *D*, and pulling each part straight.

THE CIRCUMFERENTOR.

THIS instrument is composed of a brass circular box, about five or six inches in diameter; within which is a brass ring, divided on the top into 360 degrees, and numbered 10, 20, 30, &c. to 360: in the centre of the box is fixed a steel pin finely pointed, called a centre-pin, on which is placed a needle touched by a loadstone, which always retains the same situation; that is, it always points to the north and south points of the horizon nearly, when the instrument is horizontal, and the needle at rest.

The box is covered with a glass lid in a brass rim, to prevent the needle being disturbed by wind or rain at the time of surveying: there is also a brass lid or cover, which is laid over the former to preserve the glass in carrying the instrument.

This box is fixed by screws to a brass index or ruler of about 14 or 15 inches in length, to the ends whereof are fixed brass sights which are screwed to the index and stand perpendicular thereto: in each sight is a large and a small aperture or slit, one over the other; but these are changed, that is, if the large aperture be uppermost in the one sight, it will be lowest in the other, and so of the small ones: therefore the small aperture in one is opposite to the large one in the other, in the middle of which last there is placed a horse-hair or fine silk thread.

The instrument is then fixed on a ball and socket, by the help of which and a screw you can readily fix it horizontally in any given direction, the socket being fixed on the head of a three-legged staff, whose legs, when extended, support the instrument while it is used.

To take field-notes by the Circumferentor.

PL. 6. *fig.* 6.

Let your instrument be fixed at any angle as *A*, your first station; and let a person stand at the next angle *B*, or cause a staff with a white sheet to be set there perpendicularly for an object to take your view to: then having placed your instrument horizontally (which is easily done by turning the box so that the ends of the needle may be equidistant from its bottom, and it traverses or plays freely) turn the flower-de-luce, or north part of the box, to your eye, and looking through the small aperture turn the index about till you cut the person or object in the next angle *B* with the horse-hair or thread of the opposite sight; the degrees then cut by the south end of the

needle will give the number to be placed in the second column of your field-book in a line with station No. 1, and expresses the number of degrees the stationary line is from the north, counting quite round with the sun.

Most needles are pointed at the south end, and have a small ring at the north: such needles are better than those which are pointed at each end, because the surveyor cannot mistake by counting to a wrong end, which error may be frequently committed in using a two-pointed needle.

Two-pointed needles have sometimes a ring, but more usually a cross towards the north end; and the south end is generally bearded towards its extremity, and sometimes not, but its arm is a naked right line from the cap at the centre.

Having taken the degrees or bearing of the first stationary line *AB*, let the line be measured, and the length thereof in chains and links be inserted in the third column of your field-book, under the title of distances, opposite to station No. 1.

It is customary, and even necessary, to cause a sod to be dug up at each station or place where you fix the instrument, to the end that if any error should arise in the field-book it may be the more readily adjusted and corrected, by trying over the former bearings and stationary distances.

Having done with your first station, set the instrument over the hole or spot where your object stood, as at *B*, for your second station, and send him forward to the next angle of the field, as at *C*; and having placed the instrument in a horizontal direction, with the sights directed to the object at *C*, and the north of the box next your eye, count your degrees to the south end of the needle, which register in your field-book in the second column opposite to station No. 2; then measure the stationary distance *BC*, which insert in the third column; and thus proceed from angle to angle, sending your object before you, till you return to the place where you began, and you will have the field-book complete; observing always to signify the parties' names who hold the contiguous lands, and the names of the town-lands, rivers, roads, swamps, lakes, &c. that bound the land you survey, as before; and this is the manner of taking field-notes by what is called fore-sights.

But the generality of mearsmen frequently set themselves in disadvantageous places, so as often to occasion two or more stations to be made where one may do, which creates much trouble and loss of time; we will therefore show how this may be remedied, by taking back-sights, thus: let your object stand at the point where you begin your survey, as at *A*; leaving him there, proceed to your next angle *B*, where fix your instru-

ment so that you may have the longest view possible towards *C*. Having set the instrument in a horizontal position, turn the south part of the box next your eye, and having cut your object at *A*, reckon the degrees to the south point of the needle, which will be the same as if they were taken from the object to the instrument, the direction of the index being the same. Let the degree be inserted in the field-book, and the stationary distance be measured and annexed thereto in its proper column; and thus proceed from station to station, leaving your object in the last point you left till you return to the first station *A*.

By this method your stations are laid out to the best advantage, and two men may do the business of three, for one of those who chain may be your object; but in fore-sights you must have an object before you, besides two chainmen.

It was said before, that a surveyor should have a person with him to carry the hinder end of the chain, on whom he can depend: this person should be expert and ready at taking offsets, as well as exact in giving a faithful return of the length of every stationary line. One who has such a person, and who uses back-sights, will be able to go over nearly double the ground he could in the same time by taking fore-sights, because of overseeing the chaining; for should he take back-sights he must be obliged, after taking his degree, to go back to the foregoing station, to oversee the chaining, and by this means to walk three times over every line, which is a labour not to be borne.

Or a back and a fore-sight may be taken at one station, thus: with the south of the box to your eye, observe from *B* the object *A*, and set down the degree in your field-book cut by the south end of the needle. Again, from *B* observe an object at *C*, with the north of the box to your eye, and set down the degree cut by the south point of the needle, so have you the bearings of the lines *AB* and *BC*; you may then set up your instrument at *D*, from whence take a back-sight to *C* and a fore-sight to *E*: thus the bearings may be taken quite round, and the stationary distances being annexed to them will complete the field-book.

But in this last method care must be taken to see that the sights have not the least cast on either side; if they have, it will destroy all: and yet with the same sights you may take a survey by fore-sights, or by back-sights only, with as great truth as if the sights were ever so erect, provided the same cast continues without any alteration; but, upon the whole, back-sights only will be found the readiest method.

If your needle be pointed at each end, in taking fore-sights

you may turn the north part of the box to your eye, and count your degrees to the south part of the needle, as before; or you may turn the south of the box to your eye, and count your degrees to the north end of the needle.

But in back-sights you may turn the north of the box to your eye, and count your degrees to the north point of the needle; or you may turn the south of the box to your eye, and count your degrees to the south end of the needle.

The brass ring in the box is divided on the side into 360 degrees, thus: from the north to the east into 90, from the north to the west into 90, from the south to the east into 90, and from the south to the west into 90 degrees; so the degrees are numbered from the north to the east or west, and from the south to the east or west.

The manner of using this part of the instrument is this: having directed your sights to the object, whether fore or back, as before, observe the two cardinal points of your compass the point of the needle lies between (the north, south, east, and west being called the four cardinal points, and are graved on the bottom of the box), putting down those points together by their initial letters, and thereto annexing the number of degrees, counting from the north or south, as before, thus; if the point of your needle lies between the north and east, north and west, south and east, or south and west points in the bottom of the box, then put down *NE*, *NW*, *SE*, or *SW*, annexing thereto the number of degrees cut by the needle on the side of the ring, counting from the north or south, as before.

But if the needle point exactly to the north, south, east, or west, you are then to write down *N*, *S*, *E*, or *W*, without annexing any degree.

This is the manner of taking field-notes, whereby the content of ground may be universally determined by calculation; and they are said to be taken by the quartered compass or by the four nineties.

To find the number of degrees contained in any given angle.

Set up your instrument at the angular point, and thence direct the sights along each leg of the angle, and note down their respective bearings, as before; the difference of these bearings, if less than 180, will be the quantity of degrees contained in the given angle; but if more take it from 360, and the remainder will be the degrees contained in the given angle.

Ex. Let the angle proposed be *GAB* (pl. 6, fig. 6); place the instrument at *A*, with the flower-de-luce towards you; then

direct the sights to *B*, and observe what degrees are cut by the south end of the needle, which let be 250°; then turning the instrument about on its stand, direct the sights to *G*, note again what degrees are cut by the south end of the needle, which suppose are 172°. Then 250° — 172° = 68° = the ∠ *GAB*; but if the degrees cut should be 298° and 105°, then 298°—105° =193°, which taken from 360° leaves 167° = the ∠ *GAB*.

THE THEODOLITE.

Fig. 1. *Frontispiece.*

This instrument is a circle, commonly of brass, of ten or twelve inches in diameter, whose limb is divided into 360 degrees, and those again are subdivided into smaller parts, as the magnitude of it will admit; sometimes by equal divisions and sometimes by diagonals drawn from one concentric circle of the limb to another.

In the middle is fixed a circumferentor with a needle; but this is of little or no use, except in finding a meridian line, or the proper situation of the land.

Over the brass circle is a pair of sights, fixed to a moveable index, which turns on the centre of the instrument, and upon which the circumferentor-box is placed.

This instrument will either give the angles of the field or the bearing of every stationary distance line from the meridian, as the circumferentor and quartered compass do.

To take the angles of the field.

Pl. 6. *fig.* 6.

Lay the ends of your index to 360° and 180°; turn the whole about with the 360 from you; direct the sights from *A* to *G*, and screw the instrument fast; direct them from *A* to cut the object at *B*; the degree then cut by that end of the index which is opposite you will be the quantity of the angle *GAB* to place in your field-book; to which annex the measure of the line *AB* in chains and links; set up your instrument at *B*, unscrew it, and lay the ends of your index to 360 and 180; turn the whole about, with the 360 from you or 180 next you, till you cut the object at *A*; screw the instrument fast and direct your sights to the object at *C*, and the degree then cut by that end of the index which is opposite to you will be the quantity of the angle *ABC*.

Thus proceed from station to station, still laying the index to 360, turning it from you, and observing the object at the foregoing station, screwing the instrument fast and observing the object at the following station, and counting the degrees to the opposite end of the index, will give you the quantity of each respective angle.

LEMMA.

All the angles of any polygon are equal to twice as many right angles as there are sides, less by four. Thus, all the angles A, B, C, D, E, F, G, are equal to twice as many right angles as there are sides in the figure, less by four.

PL. 6. *fig.* 6.

Let the polygon be disposed into triangles by lines drawn from any assigned point *H* within it, as by the lines *HA*, *HB*, *HC*, &c. It is evident, then (by theo. 2, sect. 4, part 1), that the three angles of each triangle are equal to two right, and consequently that the angles in all the triangles are twice as many right ones as there are sides: but all the angles about the point *H* are equal to four right (by cor. 2, theo. 1, sect. 4); therefore the remaining angles are equal to twice as many right ones as there are sides in the figure, abating four. Q. E. D.

SCHOLIUM.

Hence we may know if the angles of a survey be truly taken; for if their sum be equal to twice as many right angles as there are stations, abating four right angles, you may conclude that the angles were truly taken, otherwise not.

If you take the bearing of any line with the circumferentor, that bearing will be the number of degrees the line is from the north; consequently the north must be a like number of degrees from the line; and thus the north, and of course the south, as well as the east and west, or the situation of the land, is obtained.

To take the bearing of each respective line from the meridian; or to perform the office of the circumferentor, or quartered compass, by the theodolite.

Set your instrument at the first station, and lay the index to 360° and 180° with the flower-de-luce of the box next 360; unscrew the instrument, and turn the whole about till the north and south points of the needle cut the north and south points in the box; then screw it fast, and the instrument is north and south, if there be no variation in the needle; but if there be, and its quantity known, it may be easily allowed.

The circumferentor-box may then be taken off.

Direct the sights to the object at the second station, and the degree cut by the opposite end of the index will be the bearing of that line from the north, and the same that the circumferentor would give.

After having measured the stationary distance, set up your instrument at the second station; unscrew it, and set either end of the index to the degree of the last line, and turning the whole about with that degree towards you, direct your sights to an object at the foregoing station, and screw the instrument fast; it will then be parallel to its former situation, and consequently north and south; direct then your sights to an object at the following station, and the degree cut by the opposite end of the index will be the bearing of that line.

In the like manner you may proceed through the whole.

If the brass circle be divided into four nineties, from 360 and 180, and the letters *N*, *S*, *E*, *W* be applied to them, the bearings may be obtained by putting down the letters the far or opposite end of the index lies between, and annexing thereto the degrees from the *N* or *S*, and this is the same as the quartered compass.

If you keep the compass-box on, to see the mutual agreement of the two instruments: after having fixed the theodolite north and south, as before, turn the index about, the north end or flower-de-luce next your eye, and count the degree to the opposite or south end of the index, and this will correspond with the degree cut by the south end of the needle.

At the second or next station, unscrew the instrument and set the south of the index to the degree of the last station; turn the whole about, with the south of the index to you, and cut the object at the foregoing station; screw the instrument fast, and with the north of the index to you, cut the object at the next following station; the degree then cut by the south of the index will correspond with the degree cut by the south end of the needle, and so through the whole.

Some theodolites have a standing pair of sights fixed at 360 and 180, besides those on the moveable index; if you would use both, look through the standing sights with the 180 next you to an object at the foregoing station; screw the instrument fast, and direct the upper sights on the moveable index to the object at the following station, and the degree cut by the opposite end of the index will give you the quantity of the angle of the field.

Two pair of sights can be of no use in finding the angles from the meridian; and inasmuch as one pair is sufficient to

find the angles of the field, the second can be of no use: besides, they obstruct the free motion of the moveable index, and therefore are rather an incumbrance than of any real use.

Some will have it that they are useful with the others for setting off a right angle in taking an offset: and surely this is as easily performed by the one pair on the moveable index: thus, if you lay the index to 360 and 180, and cut the object either in the last or following station, screw the instrument fast and turn the index to 90 and 270, and then it will be at right angles with the line. So that the small sights, at those of the circle, can be of no additional use to the instrument, and therefore should be laid aside as useless.

This instrument may be used in windy and rainy weather, as well as in mountainous and hilly grounds; for it does not require a horizontal position to find the bearing or angle, as the needle doth, and therefore is preferred to any instrument that is governed by the needle.

THE SEMICIRCLE.

This instrument, as its name imports, is a half-circle, divided from its diameter into 180 degrees, and from thence again, that is, from 0 to 360 degrees. It is generally made of brass, and is from 8 to 18 inches diameter.

On the centre there is a moveable index with sights, on which is placed a circumferentor-box, as in the theodolite.

This instrument may be used as the theodolite in all respects, but with this difference; when you are to reckon the degree to that end of the index which is off the semicircle, you may find it at the other end, reckoning the degree from 180 forwards.

THE PLANE TABLE.*

A Plane Table is an oblong of oak, or other wood, about 15 inches long and 12 broad. They are generally composed of three boards, which are easily taken asunder or put together for the convenience of carriage.

* This instrument is not much used by surveyors at present.

There is a box frame, with six joints in it, to take off and put on as occasion serves; it keeps the table together, and is likewise of use to keep down a sheet of paper which is put thereon.

The outside of the frame is divided into inches and tenths, which serve for ruling parallels or squares on the paper, or for shifting it, when occasion serves.

The inside of the frame is divided into 360 degrees, which, though unequal on it, yet are the degrees of a circle produced from its centre, or centre of the table, where there is a small hole.

The degrees are subdivided as small as their distance will admit; at every tenth degree are two numbers, one the number of degrees, the other its complement, to 360.

There is another centre-hole about one-fourth of the table's breadth from one edge, and is in the middle between the two ends. To this centre-hole on the other side of the frame, there are the divisions of a semicircle, or 180 degrees; and these again are subdivided into halves, or quarters, as the size of the instrument will admit.

That side of the frame on which the 360 degrees are, supplies the place of a theodolite, the other that of a semicircle.

There is a circumferentor-box of wood, with a paper chart at the bottom, applied to one side of the table by a dovetail joint fastened by a screw. This box (besides its rendering the plane table capable of answering the end of a circumferentor) is very useful for placing the instrument in the same position every remove.

There is a brass ruler or index, about two inches broad, with a sharp or fiducial edge, at each end of which is a sight; on the ruler are scales of equal parts, with and without diagonals, and a scale of chords; the whole is fixed on a ball and socket, and set on a three-legged staff.

To take the angles of a field by the table.

Having placed the instrument at the first station, turn it about till the north end of the needle be over the meridian, or flower-de-luce of the box, and there screw it fast. Assign any convenient point, to which apply the edge of the index, so as through the sights you may see the object in the last station, and by the edge of the index from the point draw a line. Again, turn about the index with its edge to the same point, and through the sights observe the object in the second station, and from the point, by the edge of the index, draw another line; so is the angle laid down; on that last line set off the distance to the second station, in chains and links; apply your instrument

to the second station, taking the angle as before; and after the like manner proceed till the whole is finished.

This method may be used in good weather, if the needle be well touched and play freely; but if it be in windy weather, or the needle out of order, it is better, after having taken the first angle as before, and having removed your instrument to the second station, and placed the needle over the meridian line as before, to lay the index on the last drawn line, and look backward through the sights; if you then see the object in the first station, the table is fixed right, and the needle is true; if not, turn the table about, the index lying on the last line, till through the sights you see the object in the first station: and then screw it fast, and keeping the edge of the index to the second station, direct your sights to the next; draw a line by the edge of the index, and lay off the next line; and proceed through the whole without using the needle, as you do with the theodolite.

If the sheet of paper on the table be not large enough to contain the map of the ground you survey, you must put on a clean sheet, when the other is full; and this is called shifting of paper, and is thus performed.

PL. 6. *fig.* 8.

Let *ABCD* represent the sheet of paper on the plane table, upon which the plot *E*, *F*, *G*, *H*, *I*, *K*, *L*, *M* is to be drawn: let the first station be *E*; proceed as before, from thence to *F* and to *G*; then proceeding to *H*, you find there is not room on your paper for the line *GH*, however draw as much of the line *GH* as the paper can hold, or draw it to the paper's edge. Move your instrument back to the first station *E*, and proceed the contrary way to *M* and to *L*; but in going from thence to *K*, you again find your sheet will not hold it; however draw as much of the line *LK* on the sheet as it can hold.

Take that sheet off the table, first observing the distance *oo* of the lines *GH* and *LK* by the edge of the table; take off that sheet and mark it with No. 1, to signify it to be the first taken off. Having then put on another sheet, lay that distance *oo* on the contrary end of the table, and so proceed as before with the residue of the survey, from *o* to *H*, to *K*, and thence to *o*; so is your survey complete.

In the like manner you may proceed to take off and put on as many sheets as are convenient; and these may afterward be joined together with mouth glue, or fine white wafer, very thin.

If the index be fixed to the first centre, using the 360 side, it will then serve as a theodolite, and when to the second centre,

using the 180 side, it will serve as a semicircle; by either of which you may survey in rainy weather, when you cannot have paper on the table.

To measure Angles of Altitude by the Circumferentor, Theodolite, Semicircle, or Plane Table.

1. To take an angle of altitude by the circumferentor, let the glass lid be taken off, and let the instrument be turned on one side, with the stem of the ball into the notch of the socket, so that the circle may be perpendicular to the plane of the horizon; let the instrument be placed in this situation before the object, so that the top thereof may be seen through the sights; let a plummet be suspended from the centre-pin, and the object being then observed, the complement of the number of degrees comprehended between the thread of the plummet and that part of the instrument which is next your eye will give the angle of altitude required.

2. If an angle of altitude is to be taken by the theodolite, or semicircle, let a thread be run through a hole at the centre, and a plummet be suspended by it; turn the instrument on one side, by the help of the ball and notch in the socket for that purpose, so that the thread may cut 90, having 360 degrees next you; screw it fast in that position, and through the sights cut the top of the objects; and the degrees then cut by the end of the index next you are the degrees of elevation required. An angle of depression is taken the contrary way.

3. By the plane table an angle of altitude is taken in the like manner; by suspending a plummet from the centre thereof, having turned the table on one side, and fixed the index to the centre by a screw, so as to move freely, let the thread cut 90, look through the sights as before, and you have the angle of elevation, and on the contrary that of depression.

THE PROTRACTOR.

The protractor is a semicircle annexed to a scale, and is made of brass, ivory, or horn; its diameter is generally about five or six inches.

The semicircle contains three concentric semicircles, at such distances from each other that the spaces between them may contain figures.

The outward circle is numbered from the right to the left-hand, with 10, 20, 30, &c. to 180 degrees; the middlemost the same way, from 180 to 360 degrees; and the innermost from the upper edge of the scale both ways, from 10, 20, 30, &c. to 90 degrees.

It is easy to conceive that the protractor, though a semicircle, may be made to supply the place of a whole circle; for if a line be drawn, and the centre-hole of the protractor be laid on any point in that line, the upper edge of the scale corresponding with that line, the divisions on the edge of the semicircle will run from 0 to 180, from right to left: again, if it be turned the other way, or downwards, keeping the centre-hole thereof on the aforesaid point in the line, then the divisions will run from 180 to 360, and so complete an entire circle with the former semicircle.

The use of the protractor is to lay off angles, and to delineate or draw a map or plan of any ground from the field-notes; and is performed in the following manner.

To protract a field-book, when the angles are taken from the meridian.

Pl. 6. *fig.* 9.

On your paper rule lines parallel to each other, at an inch asunder (being most usual), or at any other convenient distance; on the left end of the parallels put *N* for north, and on the right *S* for south; put *E* at the top for east, and *W* at the bottom of your paper for west.

Then let the following field-book be that which is to be protracted, the bearings being taken from the meridian, whether by a circumferentor, theodolite, or semicircle, and measured with a two-pole chain.

No.	Bearing.	*ch.* *l.*
1	283½	55. 20
2	348¾	12. 36
3	317	29. 20
4	266	55. 20
5	193	40. 00
6	124	76. 00
7	63¾	87. 02

Close at the first station.

Pitch upon any convenient point on your paper for your first station, as at 1, on which lay the centre-hole of your pro-

tractor with a protracting-pin; then, if the degrees be less than 180, turn the arc of your protractor downwards, or towards the west, but if more than 180 upwards or towards the east.

Or, if the right-hand be made the north and the left the south, the west will be then up and the east down.

In this case, if the degree be less than 180, turn the arc of your protractor upwards, or towards the west; and if more, downwards, or towards the east.

By the foregoing field-book the first bearing is 283½; turn the arc of your protractor upwards, keeping the pin in the centre-hole, move the protractor so that the parallel lines may cut opposite divisions either on the ends of the scale or on the degrees, and then it is parallel. This must be always first done, before you lay off your degrees.

Then by the edge of the semicircle, keeping the protractor steady, with the pin prick the first bearing 283½, and from the centre-point, through that point or prick, draw a blank line with the pin, on which, from a scale of equal parts or from the scale's edge of the protractor, lay off the distance 55*ch.* 20*l.*; so is that station protracted.

At the end of the first station, or at 2, which is the beginning of the second, with the pin place the centre of the protractor, turning the arc up, because the bearing of the second station is more than 180, viz. 348¾. Place your protractor parallel, as before, and, by the edge of the semicircle, with the pin prick at that degree, through which and the end of the foregoing station draw a blank line, and on it set the distance of that station.

In the like manner proceed through the whole, only observe to turn the arc of your protractor down when the degrees are less than 180.

If you lay off the stationary distances by the edge of the protractor, it is necessary to observe, that if your map is to be laid down by a scale of 40 perches to an inch, every division on the protractor's edge will be one two-pole chain; ½ of a division will be 25 links, and ¼ of a division will be 12½ links.

If your map is to be laid down by a scale of 20 perches to an inch, two divisions will be one two-pole chain; one division will be 25 links; ½ a division 12½ links; and ¼ of a division will be 6¼ links.

In general, if 25 links be multiplied by the number of perches to an inch the map is to be laid down by, and the product be divided by 20 (or, which is the same thing, if you cut off one and take the half), you will have the value of one division on the protractor's edge in links and parts.

EXAMPLES.

1. How many links in a division, if a map be laid down by a scale of 8 perches to an inch?

25
8

2|0)20|0

10 links, Answer.

2. How many links in a division, if a map be laid down by a scale of 10 perches to an inch?

25
10

2|0)25|0

12.5 or 12½ links, Answer.

And so of any other.

To protract a field-book taken by the angles of the field.

Note.—We here suppose the land surveyed is kept on the right-hand as you survey.

Draw a blank line with a ruler of a length greater than the diameter of the protractor; pitch upon any convenient point therein, to which apply the centre-hole of your protractor with your pin, turning the arc upwards if the angle be less than 180, and downwards if more; and observe to keep the upper edge of the scale, or 180 and 0 degrees, upon the line: then prick off the number of degrees contained in the given angle, and draw a line from the first point through the point at the degrees, upon which lay the stationary distance. Let this line be lengthened forwards and backwards, keeping your first station to the right and second to the left, and lay the centre of your protractor over the second station, with your pin turning the arc upwards if the angle be less than 180, and downwards if more; and keeping the 180 and 0 degrees on the line, prick off the number of degrees contained in the given angle, and through that point and the last station draw a line, on which lay the stationary distance; and in like manner proceed through the whole.

In all protractions, if the end of the last station falls exactly in the point you began at, the field-work and protraction are truly

taken and performed; if not, an error must have been committed in one of them: in such case, make a second protraction; if this agrees with the former, and neither meet nor close, the fault is in the field-work and not in the protraction and then a resurvey must be taken.

REMARKS.

The accuracy of geometrical and trigonometrical mensuration depends in a great degree on the exactness and perfection of the instruments made use of; if these are defective in construction or difficult in use the surveyor will either be subject to error or embarrassed with continual obstacles. If the adjustments by which they are to be rendered fit for observation be troublesome and inconvenient, they will be taken upon trust, and the instrument will be used without examination, and thus subject the surveyor to errors that he can neither account for nor correct.

In the present state of science it may be laid down as a maxim, that every instrument should be so contrived that the observer may easily examine and rectify the principal parts; for however careful the instrument-maker may be, however perfect the execution thereof, it is not possible that any instrument should long remain accurately fixed in the position in which it came out of the maker's hand, and therefore the principal parts should be moveable, to be rectified occasionally by the observer.

An enumeration of Instruments useful to a Surveyor,

Fewer or more of which will be wanted, according to the extent of his work and the accuracy required.

A case of good pocket instruments.
A pair of beam compasses.
A set of feather-edged plotting scales.
Three or four parallel rules.
A pair of proportional compasses.
A pair of triangular ditto.
A pentagraph.
A cross-staff.
A circumferentor.
A Hadley's sextant.
An artificial horizon.
A theodolite.
A surveying compass.
Measuring chains and measuring tapes.

King's surveying quadrant.
A perambulator, or measuring wheel.
A spirit-level with telescope.
Station staves used with the level.
A protractor, with or without a nonius.

To be added for County and Marine Surveying.

An astronomical quadrant, or circular instrument.
A good refracting and reflecting telescope.
A copying-glass.

For Marine Surveying.

A station-pointer.
An azimuth compass.
One or two boat compasses.

Besides these, a number of measuring rods, iron pins, or arrows, &c. will be found very convenient, and two or three offset-staves, which are straight pieces of wood six feet seven inches long, and about an inch and a quarter square: they should be accurately divided into ten equal parts, each of which will be equal to one link. These are used for measuring offsets and to examine and adjust the chain.

Five or six staves, of about five feet in length and one inch and a half in diameter, the upper part painted white, the lower end shod with iron, to be struck into the ground, as marks.

Twenty or more iron arrows, ten of which are always wanted to use with the chain, to count the number of links, and preserve the direction of the chain, so that the distance measured may be *really* in a straight line.

The pocket measuring tapes, in leather boxes, are often very convenient and useful. They are made to the different lengths of one, two, three, four poles, or sixty-six feet and one hundred feet: divided on one side into feet and inches, and on the other into links of the chain. Instead of the latter are sometimes placed the centesimals of a yard, or three feet into 100 equal parts.

SECTION II.

MENSURATION OF HEIGHTS AND DISTANCES.

1st. *Of Heights.*

PL. 5. *fig.* 18.

THE instrument of least expense for taking heights, is a quadrant divided into ninety equal parts or degrees; and those may be subdivided into halves, quarters, or eighths, according to the radius, or size of the instrument: its construction will be evident by the scheme thereof.

From the centre of the quadrant let a plummet be suspended by a horse-hair, or a fine silk thread, of such a length that it may vibrate freely, near the edge of its arc; by looking along the edge AC, to the top of the object whose height is required, and holding it perpendicular, so that the plummet may neither swing from it, nor lie on it, the degree then cut by the hair or thread will be the angle or altitude required.

If the quadrant be fixed upon a ball and socket on the three-legged staff, and if the stem from the ball be turned into the notch of the socket, so as to bring the instrument into a perpendicular position, the angle of altitude by this means can be acquired with much greater certainty.

An angle of altitude may be also taken by any of the instruments used in surveying; as has been particularly shown in treating of their description and uses.

Most quadrants have a pair of sights fixed on the edge AC, with small circular holes in them; which are useful in taking the sun's altitude, requisite to be known in many astronomical cases; this is effected by letting the sun's ray, which passes through the upper sight, fall upon the hole in the lower one; and the degree then cut by the thread will be the angle of the sun's altitude; but those sights are useless for our present purpose,—for looking along the quadrant's edge to the top of the object will be sufficient as before.

To take an angle of altitude and depression with the quadrant.

PL. 14. *fig.* 6, 7.

Let A be any object, as the sun, moon, or a star, or the top of a tower, hill, or other eminence: and let it be required to find the angle ABC, which a line drawn from the object makes above the horizontal line BC.

Place the centre of the quadrant in the angular point, and

move it round there as a centre, till with one eye at *D*, the other being shut, you perceive the object *A* through the sights; then will the arc *GH* of the quadrant, cut off by the plumb-line *BH*, be the measure of the angle *ABC*, as required.

The angle *ABC* of depression of any object *A*, below the horizontal line *BC*, is taken in the same manner; except that here the eye is applied to the centre, and the measure of the angle is the arc *GH*, on the other side of the plumb-line.*

Demonstration. In taking the angle of altitude, the angle *ABG* is a right angle; and because the plumb-line *BH* is perpendicular to the horizon, the angle *CBH* is also a right angle; hence if the angle *CBG* be taken from these equals, the remaining angles will be equal, that is *ABC*=*GBH*, or equal to the arc *HG*. Q. E. D.

In like manner, the angle *GBH* (in taking the angle of depression) is equal to the angle *ABC*.

PROBLEM I.

PL. 5. *fig.* 19.

To find the height of a perpendicular object at one station, which is on a horizontal plane.

A steeple.

Given
- The angle of altitude, 53 degrees.
- Distance from the observer to the foot of the steeple, or the base, 85 feet.
- Height of the instrument, or of the observer, 5 feet.

Required the height of the steeple.

The figure is constructed and wrought, in all respects, as Case 2 of right-angled trigonometry; only there must be a line drawn parallel to and beneath *AB*, of 5 feet, for the observer's height, to represent the plane upon which the object stands; to which the perpendicular must be continued, and that will be the height of the object.

Thus, *AB* is the base, *A* the angle of altitude, *BC* the height of the steeple from the instrument, or from the observer's eye,

* In finding the height of an object, let the observed angle be as near 45° as possible, for then a small error committed in taking it makes the least error in the computed height of the object. In taking the height of a perpendicular object, if the observed angle be 45°, the height of the object above the horizontal line is equal to the base line, and if the observed angle should be 60°, three times the square of the base line is equal to the square of the perpendicular object above the horizontal line; hence by extracting the square root of three times the square of the base or horizontal line, will give the height of the object above that line, to which add the height of the observer's eye above the horizon, and you have the true height.

if he were at the foot of it; DC the height of the steeple above the horizontal surface.

Various statings for BC, as in Case 2 of right-angled plane trigonometry.

$$90°$$
$$53 = A.$$
$$37 = C.$$

1. $S.C : AB : : S.A : BC.$
   $37°\ \ 85\ \ 53°\ \ 112.8.$
2. $R. : AB : : T.A : BC.$
   $90°\ \ 85\ \ 53°\ \ 112.8.$
3. $T.C : AB : : R. : BC.$
   $37°\ \ 85\ \ 90°\ \ 112.8.$

To BC 112.8
Add DB 5. the height of the observer.

Their sum is 117.8 or 118 feet, the height of the steeple required.

PROBLEM II.

PL. 5. *fig.* 20.

To find the height of a perpendicular object, on a horizontal plane, by having the length of the shadow given.

Provide a rod, or staff, whose length is given, let that be set perpendicular, by the help of a quadrant, thus; apply the side of the quadrant AC to the rod, or staff; and when the thread cuts 90°, it is then perpendicular; the same may be done by a carpenter's or mason's plumb.

Having thus set the rod or staff perpendicular, measure the length of its shadow when the sun shines, as well as the length of the shadow of the object whose height is required, and you have the proper requisites given. Thus,

ab, the length of the shadow of the staff, 15 feet.
bc, the length of the staff, 10 feet.
AB, the length of the shadow of the steeple, or object, 135 feet.
Required BC, the height of the object.

The triangles abc, ABC are similar, thus; the angle $b = B$, being both right; the lines ac, AC are parallel, being rays, or a ray of the sun; whence the angle $a = A$ (by part 3, theo. 3, sect. 4), and consequently $c = C$. The triangles, being therefore mutually equiangular, are similar (by theo. 16, sect. 4), it will be,

$ab : bc : : AB : BC.$
15 10 135 90, the steeple's height required.

The foregoing method is most to be depended on; however, this is mentioned for variety's sake.

PROBLEM III.

Pl. 5. *fig.* 21.

To take the altitude of a perpendicular object at the foot of a hill, from the hill's side.

Turn the centre A of the quadrant next your eye, and look along the side AC, or 90 side, to the top and bottom of the object; and noting down the angles, measure the distance from the place of observation to the foot of the object. Thus,

Given { Angle to the foot of the object, $55\frac{1}{4}°$ or 55° 15′.
Angle to the top of it, $31\frac{1}{4}°$ or 31° 15′
Distance to the foot of it, 250 feet.

Required the height of the object.

By Construction.

Draw an indefinite blank line AD at any point in which A makes the angles EAB of 55° 15′, and EAC of 31° 15′; lay 250 from A to B; from B draw the perpendicular BE (by prob. 7 of geometry) crossing AC in C: so will BC be the height of the object required.

In the triangle ABC there is given,

ABE the complement of EAB to 90°, which is 34° 45′.

CAB the difference of the given angles 24° 00′.

The side AB 250. Required BC.

This is performed as Case 2 of oblique angular trigonometry. Thus,

180 — the sum of ABE 34° 45′ and CAB 24° 00′ = ACB 121° 15′. Then,

$S.ACB : AB :: S.CAB : BC.$

121° 15′ 250 24° 00′ 119, the height required.

PROBLEM IV.

Pl. 5. *fig.* 22.

To take the altitude of a perpendicular object on the top of a hill at one station, when the top and bottom of it can be seen from the foot of the hill.

As in prob. 1, take an angle to the top and another to the bottom of the object, and measure from the place of observation to the foot of the object, and you have all the given requisites. Thus,

A tower on a hill.

Given { Angle to the bottom 48° 30′.
Angle to the top 67° 00′.
Distance to the foot of the object 136 feet.

Required the height of the object.

By Construction.

Make the angle DAB=48° 30′, and lay 136 feet from A to B; from B let fall the perpendicular BD, and that will be the height of the hill. Produce BD upwards by a blank line. Again, at A make the angle DAC=67° 00′ by a blank line, and from C, where that crosses the perpendicular produced, draw the line CB, and that will be the height of the object required.

Let AC be drawn.

In the triangle ABC there is given,

The angle ACD the complement of DAC=23° 00′.

CAB the difference between the two given angles = 18° 30′.

And the side AB 136. To find BC,

$$S.C : AB :: S.CAB : BC.$$
$$23° \quad 136 \quad 18° 30′ \quad 110\tfrac{1}{2}$$

If BD were wanted it is easily obtained by the first case of right-angled plane trigonometry.

PROBLEM V.

Pl. 5. *fig.* 23.

To take an inaccessible perpendicular altitude on a horizontal plane.

This is done at two stations. Thus,

Let DC be a tower which cannot be approached, by means of a moat or ditch, nearer than B; at B take an angle of altitude to C: measure any convenient distance backward to A, which note down; at A take another angle to C; so have you the given requisites. Thus,

Given { First angle 55° 00′.
Stationary distance 87 feet.
Second angle 37° 00′.

The height of the tower CD is required.

By Construction.

Upon an indefinite blank line lay off the stationary distance 87 from A to B; from B set off your first, and from A your

second angle; from *C*, the point of intersection of the lines which form these angles, let fall the perpendicular *CD*; and that will be the height of the object required.

The external angle *CBD* of the triangle *ABC* is equal to the two internal opposite ones *A* and *ACB* (by theo. 4, sect. 4): wherefore if one of the internal opposite angles be taken from the external angle, the remainder will be the other internal opposite one. Thus,

CBD 55°—*A* 37°=*ACB* 18°.

Therefore, in the triangle *ABC* we have the angles *A* and *ACB* with the side *AB* given, to find *BC*.

S.ACB : *AB* : : *S.A* : *BC*.
18° 87 37° 169.4.

Having found *BC*, we have in the triangle *BCD* the angle *CBD* 55°; consequently *BCD* 35°, and *BC* 169.4, to find *DC*.

This is performed by the first case of right-angled trigonometry three several ways. Thus:

1. *R* : *BC* : : *S.CBD* : *DC*.
90° 169.4 55° 138.8,
the height required.

2. Sec. *CBD* : *BC* : : *T.CBD* : *DC*.
55° 169.4 55° 138.8,
the height required.

3. Sec. *BCD* : *BC* : : *R* : *CD*.
35° 169.4 90° 138.8,
the height required.

If *BD*, the breadth of the moat, were required, it may also be found by three different statings, as in the first case of right-angled plane trigonometry.

PROBLEM VI.

Pl. 5. *fig.* 24.

Let *BC*, a maypole, whose height is 100 feet, be broken at *D*, the upper part of which, *DC*, falls upon a horizontal plane, so that its extremity *C* is 34 feet from the bottom or foot of the pole.

Required the segments *BD* and *DC*.

By Construction.

Lay 34 feet from *A* to *B*; on *B* erect the perpendicular *BC* of 100 feet, and draw *AC*; bisect *AC* (by prob. 4, geom.)

with the perpendicular line EF; and from D, where it cuts the perpendicular BC, draw AD, which will be the upper segment, and DB will be the lower.

By cor. to lemma preceding theo. 7, geom., $AD=DC$; and (by the lemma) the angle $C=CAD$.

In the triangle ABC find C, as in Case 6 of right-angled trigonometry. Thus,

1. $BC : R : : AB : T.C=GAD.$
   100 90° 34 18° 47′.

By theo. 4, geom. The external angle $ADB=37°\ 34'$, or to twice the angle C; *i. e.* to C and GAD.

Then in the triangle ABD there is ADB 37° 34′, therefore also its complement DAB 52° 26′ and AB 34 given, to find AD and BD.

By the second case of right-angled trigonometry:

2. $S.ADB : AB : : R : AD$ or $DC.$
   37° 34′ 34 90° 55.77.

$BC—DC=BD.$
100—55.77=44.23, required.

These may be had from other statings, as in the second case aforesaid.

PROBLEM VII.

PL. 5. *fig.* 25.

To take the altitude of a perpendicular object on a hill, from a plane beneath it.

This is done at two stations. Thus,

Let the height DC of a windmill on a hill be required.

From any part of the plane whence the foot of the object can be seen, let angles be taken to the foot and top; measure thence any convenient distance towards the object, and at the end thereof take another angle to the top, and you have the proper requisites. Thus,

First station. Angle to the foot DAB 21° 00′.
Angle to the top CAB 35° 00′.
Stationary distance AB 104 feet.

Second station. Angle to the top 48° 30′.
DC required.

By Construction.

On an indefinite blank line lay the stationary distance AB 104 feet; from A set off the second, and from B the third given angle; and from the intersecting point C of the line formed by

them let fall the perpendicular CE; from A set off the first angle, and the line formed by it will determine the point D. Thus have we the height of the hill, as well as that of the windmill.

The angle $CBE - A = ACB$, as in the last prob.

In the triangle ABC find AC thus,

$S.ACB : AB :: S.ABC$ (or sup. of CBE) $: AC.$

$13° 30' : 104 :: 131° 30' : 333.6.$

The angle $CAE - DAE = CAD.$

The angle $ACD = AED + EAD$, by theo. 4.

In the triangle CAD find CD thus,

$S.ADC : AC :: S.CAD : DC.$

$111° : 333.6 :: 14° : 86.46$ required.

CE, BE, or DE may be found by other various statings, as set forth in the first and second cases of right-angled trigonometry.

PROBLEM VIII.

Pl. 5. *fig.* 26.

To find the length of an object that stands obliquely on the top of a hill, from a plane beneath.

Let CD be a tree whose length is required.

This is done at two stations.

Make a station at B, from whence take an angle to the foot and another to the top of the tree; measure any convenient distance backward to A, from whence also let an angle be taken to the foot and another to the top, and you have the requisites given. Thus:

First station. Angle to the foot $EBD = 36° 30'$.
Angle to the top $EBC = 44° 30'$.
Stationary distance $AB = 104$ feet.

Second station. Angle to the foot $EAD = 24° 30'$.
Angle to the top $EAC = 32° 00'$.

Let DC and DE be required.

The geometrical constructions of this and the next problem are omitted, as what has been already said, and the figures, are looked upon as sufficient helps.

$EBC - A = ACB$, or $44° 30' - 32° = 12° 30'$, as before.

In the triangle ABC find BC thus,

1. $S.ACB : AB :: S.A : BC.$

$12° 30' \quad 104 \quad 32° \quad 254.7.$

$EBD - EAD = ADB$, or $36° 30' - 24° 30' = 12° 00'$.

In the triangle *ADB* find *DB* thus,

2. *S.ADB* : *AB* : : *S.DAB* : *DB*.
12° 00′ 104 24° 30′ 207.4.

CBE—*DBE*=*CBD*, or 44° 30′—36° 30′=8° 00′.

In the triangle *CBD* there is given *CB* 254.7, *DB* 207.4, and the angle *CBD* 8° 00′, to find *DC*.

This is performed as Case 3 of oblique-angled trigonometry. Thus,

3. *BC*+*BD* : *BC*—*BD* : : *T.* of ½(*BDC*+*BCD*) : *T.* of
462.1 47.3 86° 00′
½(*BDC*—*BCD*).
55° 40′.

86° 00′+55° 40 =141° 40′=*BDC*.
86° 00′—55° 40′= 30° 20′=*BCD*.

4. *S.BCD* : *BD* : : *S.CBD* : *DC*.
30° 20′ 207.4 8° 00′ 57.15, length of the tree.

To find *DE* in the triangle *DBE*, say

R. : *BD* : : *S.DBE* : *DE*.
90° 207.4 36° 30′ 123.4, height of the hill.

PROBLEM IX.

To find the height of an inaccessible object CD, on a hill BC, from ground that is not horizontal.

Pl. 6. *fig.* 1.

From any two points, as *G* and *A*, whose distance *GA* is measured, and therefore given; let the angles *HGD*, *BAD*, *BAC*, and *EAG* be taken; because *GH* is parallel to *EA* (by part 2, theo. 3, geom.), the angle *HGA*=*EAG*; therefore *EAG*+*HGD*=*AGD*: and (by cor. 1, theo. 1, geom.) 180 — the sum of *EAG* and *BAD*=*GAD*; and, by cor. 1, theo. 5, geom., 180 — the sum of the angles *AGD* and *GAD*=*GDA*: thus we have the angles of the triangle *AGD* and the side *AG* given; thence (by Case 2 of obl. ang. trig.) *AD* may be easily found. The angle *DAB*—*CAB*=*DAC*, and 90°—*BAD*=*ADC*, and 180° — the sum of *DAC* and *ADC*=*ACD*: so have we the several angles of the triangle *ACD* given, and the side *AD*; whence (by Case 2 of obl. trig.) *CD* may be easily found. We may also find *AC*, which with the angle *BAC* will give *CB* the height of the hill.

The solutions of the several problems in heights and distances by Gunter's scale are omitted; because every particular stating has been already shown by it, in Trigonometry.

2d. *Of Distances.*

The principal instruments used in surveying will give the angles or bearings of lines; which was particularly shown when we treated of them.

PROBLEM I.

Pl. 6. *fig.* 2.

Let A and B be two houses on one side of a river, whose distance asunder is 293 perches: there is a tower at C on the other side of the river, that makes an angle at A with the line AB of 53° 20′, and another at B with the line BA of 66° 20′; required the distance of the tower from each house, viz. AC and BC.

This is performed as Case 2 of oblique-angled trigonometry, thus,

1. $S.C : AB :: S.A : BC.$
   60° 20′ 293 53° 20′ 270.5.
2. $S.C : AB :: S.B : AC.$
   60° 20′ 293 66° 20′ 308.8.

PROBLEM II.

Pl. 6. *fig.* 11.

Let B and C be two houses whose direct distance asunder, BC, is inaccessible: however, it is known that a house at A is 252 perches from B, and 230 from C, and that the angle BAC is found to be 70°. What is the distance BC between the two houses?

This is performed as Case 3 of oblique-angled trigonometry, thus,

1. $AB+AC : AB—AC :: T.$ of $\frac{1}{2}(C+B) : T.$ of $\frac{1}{2}(C—B).$
   482 22 55° 00′ 3° 44′.

   $55°+3°\ 44′=58°\ 44′=C.$ $55°—3°\ 44′=51°\ 16′=B.$

2. $S.C : AB :: S.A : BC.$
   58° 44′ 252 70° 277.

PROBLEM III.

Pl. 6. *fig.* 3.

Suppose ABC a triangular piece of ground, which by an old survey we find to be thus; AB 260, AC 160, BC 150 perches,

the mearing lines *AC* and *BC* are destroyed or ploughed down, and the line *AB* only remaining. What angles must be set off at *A* and *B* to run new mearings by exactly where the old ones were?

This is performed as in Case 4 of oblique-angled trigonometry, thus,

1. *AB* : *AC*+*BC* : : *AC*—*BC* : *AD*—*DB*.
   260 310 10 11.92.
   130+5.96=135.96=*AD*.
   130—5.96=124.04=*DB*.
2. *AD* : *R* : : *AC* : sec. *A*.
   136 90° 160 31° 47′.
3. *BC* : *S.A* : : *AC* : *S.B*.
   150 31° 47′ 160 34° 10′.

PROBLEM IV.

Pl. 6. *fig.* 4.

Let *D* and *C* be two trees in a bog, to which you can have no nearer access than at *A* and *B*; there is given *DAB* 100°, *CAB* 36° 30′, *CBA* 121°, *DBA* 49°, and the line *AB* 113 perches. Required the distance of the trees *DC*.

180° — the sum of *DBA* and *DAB*=*ADB*=31°.
180° — the sum of *CAB* and *CBA*=*ACB*=22° 30′.

In the triangle *ABD*, find *DB* thus,

1. *S.ADB* : *AB* : : *S.DAB* : *DB*.
   31° 113 100° 216.

And in the triangle *ABC*, find *BC* thus,

2. *S.ACB* : *AB* : : *S.CAB* : *BC*.
   22° 30′ 113 36° 30′ 175.6.

In the triangle *DBC*, you have *DBC*=*ABC*—*ABD*=72°, likewise the sides *BD*, *BC* as before found, given, to find *DC*.

3. *BD*+*BC* : *BD*—*BC* : : *T.* of ½(*DCB*+*CDB*) : *T.* of ½(*DCB*—*CDB*).
   391.6 40.4 54°
   8° 05′.

54°+8° 05′=62° 05′=*DCB*.
54°—8° 05′=45° 55′=*CDB*.

4. *S.CDB* : *BC* : : *S.DBC* : *DC*.
   45° 55′ 175.6 72° 232.5.

LEMMA.

PL. 6. *fig.* 10.

If from a point C of a triangle ABC, inscribed in a circle, there be a perpendicular CD let fall upon the opposite side AB, that perpendicular is to one of the sides, including the angle, as the other side, including the angle, is to the diameter of the circle, i. e. $DC : AC :: CB : CE$.

Let the diameter CE be drawn, and join EB; it is plain, the angle $CEB = CAB$ (by cor. 2, theo. 7, geom.), and CBE is a right angle (by cor. 5, theo. 7, geom.), and $= ADC$: whence $ECB = ACD$. The triangles CEB, CAD are therefore mutually equiangular, and (by theo. 16, geom.) $DC : AC :: CB : CE$, or $DC : CB :: AC : CE$. Q. E. D.

PROBLEM V.

PL. 6. *fig.* 5.

Let three gentlemen's seats A, B, C be situate in a triangular form: there is given, AB 2.5 miles, AC 2.3, and BC 2. It is required to build a church at E, that shall be equidistant from the seats A, B, C. What distance must it be from each seat, and by what angle may the place of it be found?

By Construction.

By prob. 15, geom., find the centre of a circle that will pass through the points A, B, C, and that will be the place of the church; the measure of which, to any of these points, is the answer for the distance: draw a line from any of the three points to the centre, and the angle it makes with either of the sides that contain the angle it was drawn to; that angle laid off by the direction of an instrument, on the ground, and the distance before found, being ranged thereon, will give the place of the church required.

By Calculation.

1. $AB : AC+BC :: AC-BC : AD-DB$.
   2.5 4.3 .3 .516.

$$1.25+.258=1.508=AD.$$

By cor. 2, theo. 14, geom., the square root of the difference of the squares of the hypothenuse AC, and given leg AD, will give DC.

That is, $5.29-2.274064=3.015936$.

Its square root is 1.736 = *CD*.

Then, by the preceding lemma,

2. *CD* : *AC* : : *CB* : the diameter.
1.736 2.3 2 2.65.

the half of which, viz. 1.325, is the semidiameter, or distance of the church from each seat, that is, *AE*, *CE*, *BE*.

From the centre *E* let fall a perpendicular upon any of the sides as *EF*, and it will bisect it in *F* (by theo. 8, geom.).

Wherefore, $AF = CF = \frac{1}{2}AC = 1.15$.

In the right-angled triangle *AFE* you have *AF* 1.15, and *AE* the radius 1.325 given, to find *FAE*. Thus:

3. *AF* : *R* : : *AE* : sec. *FAE*.
1.15 90° 1.325 29° 47′.

Wherefore, directing an instrument to make an angle of 29° 47′ with the line *AC*, and measuring 1.325 on that line of direction, will give the place of the church, or the centre of a circle that will pass through *A*, *B*, and *C*.

The above angle *FAE* may be had without a secant, as before. Thus:

AE : *R* : : *AF* : *S.AEF*.
1.325 90° 1.15 60° 13′.

Its complement 29° 47′ will give *FAE*, as before.

PRACTICAL QUESTIONS.

Ex. 1. At 170 feet distance from the bottom of a tower the angle of its elevation was found to be 52° 30′. Required the altitude of the tower. Ans. 221.55 feet.

Ex. 2. From the top of a tower, by the seaside, of 14.3 feet high, it was observed that the angle of depression of a ship's bottom, then at anchor, measured 35°. What then was the ship's distance from the bottom of the wall?

Ans. 204.22 feet.

Ex. 3. From a window near the bottom of a house which seemed to be on a level with the bottom of a steeple, I took the angle of elevation of the top of the steeple, equal 40°; then from another window, 18 feet directly above the former, the like angle was 37° 30′. What then is the height and distance of the steeple?

Ans. { height 57.26.
distance 150.50.

Ex. 4. Wanting to know the height of an inaccessible tower, at the least distance from it, on the same horizontal plane, I

took its angle of elevation, equal to 58°; then going 300 feet directly from it, found the angle there to be only 32°. Required its height and my distance from it at the first station.

Ans. { height 307.53. distance 192.15.

Ex. 5. Being on the side of a river, and wanting to know the distance to a house which was seen on the other side, I measured out for a base 400 yards in a right line by the side of the river, and found that the two angles, one at each end of this line, subtended by the other end and the house, were 68° 2′ and 73° 15′. What then was the distance between each station and the house?

Ans. { 593.08 612.38 } yards.

Ex. 6. Wanting to know the breadth of a river, I measured a base of 500 yards in a straight line close by one side of it, and at each end of this line I found the angles subtended by the other end and a tree close to the bank on the other side of the river to be 53° and 73° 15′. What then was the perpendicular breadth of the river? Ans. 529.48 yards.

Ex. 7. Two ships of war, intending to cannonade a fort, are, by the shallowness of the water, kept so far from it that they suspect their guns cannot reach it with effect. In order therefore to measure the distance, they separate from each other a quarter of a mile, or 440 yards; then each ship observes and measures the angle which the other ship and the fort subtend, which angles are 83° 45′ and 85° 15′. What then is the distance between each ship and the fort?

Ans. { 2292.26 2298.05 } yards.

Ex. 8. A point of land was observed by a ship at sea to bear east by south; and after sailing north-east 12 miles, it was found to bear south-east by east. It is required to determine the place of that headland, and the ship's distance from it at the last observation. Ans. 26.0728 miles.

Ex. 9. If the height of the mountain called the Peak of Teneriffe be $2\frac{1}{3}$ miles, as it is very nearly, and the angle taken at the top of it, as formed between a plumb-line and a line conceived to touch the earth in the horizon, or farthest visible point, be 88° 2′; it is required from these measures to determine the magnitude of the whole earth, and the utmost distance that can be seen on its surface from the top of the mountain, supposing the form of the earth to be perfectly globular.

Ans. { distance 135.943 diameter 7916 } miles.

Ex. 10. Wanting to know the extent of a piece of water, or distance between two headlands, I measured from each of them to a certain point inland, and found the two distances to be 735 yards and 840 yards; also the horizontal angle subtended between these two lines was 55° 40′. What then was the distance required? Ans. 741.2 yards.

Ex. 11. Wanting to know the distance between a house and a mill which were seen at a distance on the other side of a river, I measured a base line along the side where I was of 600 yards, and at each end of it took the angles subtended by the other end and the house and mill, which were as follows, viz. at one end the angles were 58° 20′ and 95° 20′, and at the other end the like angles were 53° 30′ and 98° 45′. What then was the distance between the house and mill?

Ans. 962.5866 yards.

Ex. 12.* Wanting to know my distance from an inaccessible object 0 on the other side of a river, and having no instrument for taking angles, but only a chain or cord for measuring distances; from each of two stations, *A* and *B*, which were taken at 500 yards asunder, I measured, in a direct line from the object 0, 100 yards, viz. *AC* and *BD*, each equal 100 yards; also the diagonal *AD* measured 550 yards, and the diagonal *BC* 560. What then was the distance of the object 0 from each station *A* and *B*?

Ans. $\left\{\begin{matrix} A\ 0\ 526.81. \\ B\ 0\ 500.47. \end{matrix}\right\}$ yards.

SECTION III.

MENSURATION OF AREAS,

OR THE VARIOUS METHODS OF CALCULATING THE SUPERFICIAL CONTENTS OF ANY FIELD.

Definition.

THE area or contents of any plane surface in perches is the number of square perches which that surface contains.

* These practical examples are taken from Hutton's Mathematics, vol. ii. seventh London edition.

PL. 7. *fig.* 1.

Let *ABCD* represent a rectangular parallelogram, or oblong; let the side *AB* or *DC* contain eight equal parts, and the side *AD* or *BC* three of such parts; let the line *AB* be moved in the direction of *AD* till it has come to *EF*, where *AE* or *BF* (the distance of it from its first situation) may be equal to one of the equal parts. Here it is evident that the generated oblong *ABEF* will contain as many squares as the side *AB* contains equal parts, which are eight; each square having for its side one of the equal parts into which *AB* or *AD* is divided. Again, let *AB* move on till it comes to *GH*, so as *GE* or *HF* may be equal to *AE* or *BF*; then it is plain that the oblong *AGHB* will contain twice as many squares as the side *AB* contains equal parts. After the same manner it will appear that the oblong *ADCB* will contain three times as many squares as the side *AB* contains equal parts; and, in general, that every rectangular parallelogram, whether square or oblong, contains as many squares as the product of the number of equal parts in the base multiplied into the number of the same equal parts in the height contains units, each square having for its side one of the equal parts.

Hence arises the solution of the following problems.

PROBLEM I.

To find the contents of a square piece of ground.

1. Multiply the base in perches into the perpendicular in perches, the product will be the contents in perches; and because 160 perches make an acre, it must thence follow that

Any area, or contents in perches, being divided by 160, will give the contents in acres; the remaining perches, if more than 40, being divided by 40, will give the roods, and the last remainder, if any, will be perches.

Or thus:

2. Square the side in four-pole chains and links, and the product will be square four-pole chains and links: divide this by 10, or cut off one more than the decimals, which are five in all, from the right towards the left: the figures on the left are acres; because 10 square four-pole chains make an acre, and the remaining figures on the right are decimal parts of an acre. Multiply the five figures to the right by four, cutting five figures

from the product, and if any figure be to the left of them it is a rood, or roods; multiply the last cut off figures by 40, cutting off five, or (which is the same thing) by 4, cutting off four; and the remaining figures to the left, if any, are perches.

1. The first part is plain, from considering that a piece of ground in a square form, whose side is a perch, must contain a perch of ground; and that 40 such perches make a rood, and four roods an acre; or, which is the same thing, that 160 square perches make an acre, as before.

2. A square four-pole chain (that is, a piece of ground four poles or perches every way) must contain 160 square perches; and 160 perches make an acre; therefore 10 times 16 perches, or 10 square four-pole chains, make an acre.

Note.—The chains given or required, in any of the following problems, are supposed to be two-pole chains, that chain being most commonly used; but they must be reduced to four-pole chains or perches for calculation, because the links will not operate with them as decimals.

EXAMPLES.

PL. 1. *fig.* 17.

Let *ABCD* be a square field, whose side is 14*ch.* 29*l.*; required the contents in acres.

By problem 4, section 1, part 2, 14*ch.* 29*l.* are equal to

29.16 perches
29.16

17496
2916
26244
5832

160)850.3056(A. R. P.
5 1 10, contents.

40)50(1 rood.
10 perches.

Or thus:

14*ch.* 29*l.* are equal to 7*ch.* 29*l.* of four-pole chains, by problem 1, section 1, part 2.

ch. l.
7.29
7.29

6561
1458
5103

			A.	R.	P.
Acres 5	31441	contents, as before,	5	1	10.
	4				
Rood 1	25764				
	40				
Perches 10	30560				

It is required to lay down a map of this piece of ground, by a scale of twenty perches to an inch.

Take 29.16, the perches of the given side, from the small diagonal on the common surveying scale, where twenty small, or two of the large divisions are an inch: make a square whose side is that length (by prob. 9, geom.), and it is done.

PROBLEM II.

To find the side of a square whose contents are given.

Extract the square root of the given contents in perches, and you have the side in perches, and consequently in chains.

EXAMPLE.

It is required to lay out a square piece of ground which shall contain 12A. 3R. 16P. Required the number of chains in each side of the square; and to lay down a map of it by a scale of 40 perches to an inch.

A.	R.	P.
12	3	16
4		
51		
40		
2056		

2056(45.34+ perches = 22*ch.* 33¼*l.* by prob. 6.
sect. 1, part 2.
85)456
903)3100
9064)39100 &c.

To draw the map.

From a scale where 4 of the large or 40 of the small divisions are an inch, take 45.34 the perches of the side, of which make a square.

PROBLEM III.

To find the contents of an oblong piece of ground.

Multiply the length by the breadth, for the contents.

EXAMPLE.

PL. 1. *fig.* 3.

Let *ABCD* be an oblong piece of ground, whose length *AB* is 14*ch.* 25*l.* and breadth 8*ch.* 37*l.* Required the contents in acres, and also to lay down a map of it, by a scale of 20 perches to an inch.

ch. l. perches.
14.25=29.00
8.37=17.48
} By prob. 4, sect. 1, part 2.

15732
3496

A. R. P.
160)506.9200(3 0 27 contents.

26 perches, or near 27.

Or thus:
four-pole ch.
ch. l. ch. l.
14.25=7.25
8.37=4.37
} By prob. 1, sect. 1, part 2.

5075
2175
2900

31682

Acres 3|16825
4

Rood |67300
4

Perches 26|9200

To draw the map.

Make an oblong (by schol. to prob. 9, geom.) whose length, from a scale of 20 to an inch, may be 29 perches, and breadth 17.48 perches.

PROBLEM IV.

The contents of an oblong piece of ground and one side given, to find the other.

Divide the contents in perches by the given side in perches, the quotient is the side required in perches; and thence it may be easily reduced to chains.

EXAMPLE.

There is a ditch 14*ch.* 25*l.* long, by the side of which it is required to lay out an oblong piece of ground which shall contain 3A. 0R. 27P. What breadth must be laid off at each end of the ditch to enclose the 3A. 0R. 27P.?

A. R. P.
3 0 27
4

12
40

perch. ch. l.
29)507(17.48=8 37, breadth.
217

140

240

8

The map is constructed like the last.

PROBLEM V.

To find the contents of a piece of ground in form of an oblique angular parallelogram, or of a rhombus or rhomboides.

RULE I.

Multiply the base into the perpendicular height. The reason is plain from theo. 13, geom.

EXAMPLE.

PL. 7. *fig.* 2.

Let *ABCD* be a piece of ground in form of a rhombus, whose base *AB* is 22 chains, and perpendicular *DE* or *FC* 20 chains. Required the contents.

ch.
22=11.0
20=10.0 } four-pole chains.

Acres 11|0

ch. Or,
22=44
20=40 } perches.

160)1760(11 acres.
160
0

The converse of this is done by prob. 4, and the map is drawn by laying off the perpendicular on that part of the base from whence it was taken, joining the extremity thereof to that of the base by a right line, and thence completing the parallelogram.

RULE II.

As rad. (viz. S. of 90°, or tang. of 45°)
Is to the sine of any angle of a parallelogram,
So is the product of the sides including the angle:
To the area of the parallelogram.

That is, $DA \times AB \times$ nat. sine of the angle A = the area.* Pl. 7, fig. 2.

* Demonstration. For, having drawn the perpendicular DE the area by the first rule is $AB \times DE$; but as radius 1 ($S. \angle E$) : $S. \angle A$: : AD :

EXAMPLE.

How many acres are in a rhomboides whose less angle is 30° and the including sides 25.35 and 10.4 four-pole chains?

Ans. 13A. 0R. 29.12P.

(Rad.) 1 : .500000 (Nat. S. of 30°) : : 263.640 (=25.35 × 10.4) : 131.82 = the area in four-pole chains; which divided by 10 (because 10 square chains are an acre) gives 13.182 acres, or 13A. 0R. 29.12P.

Note.—Because the angle of a square and rectangle are each 90°, whose sine is 1, this rule for them is the same as the first.

PROBLEM VI.

To find the contents of a triangular piece of ground.

Multiply the base by half the perpendicular, or the perpendicular by half the base; or take half the product of the base into the perpendicular.

The reason of this is plain from cor. 2, theo. 12, geom.

EXAMPLE.

PL. 1. *fig.* 16.

Let ABC be a triangular piece of ground whose longest side or base BC is 24*ch.* 38*l.*, and perpendicular AD, let fall from the opposite angle, is 13*ch.* 28*l.* Required the contents.

	ch.	*l.*	*ch.*	*l.*	
1. Base	24.	38	= 12.	38	} four-pole chains.
½ perp.			3.	39	

11142
3714
3714

Acres 4 | 19682
4

Rood | 78728
40

Perches 31 | 49120

Contents, 4A. 0R. 31P.

$DE = S.\angle A \times DA$; therefore, $DE \times AB = AB \times S.\angle A \times DA$ = the area; or, 1 : $S.\angle A$: : $DA \times AB$: $S.\angle A \times DA \times AB$ = the area of the parallelogram. Q. E. D.

ch. l. ch. l.
Perp. 13.28=6.78
½ perp. 6.39=3.39 } four-pole chains, by prob. 1, sect. 1, part 2.

Or, 2dly. Perp. 6.78 of four-pole chains.
½ base 6.19

6102
678
4068

4|19682=4A. 0R. 31P.

ch. l.
Or, 3dly. Base 12.38 four-pole chains.
Perp. 6.78

9904
8666
7428

83.9364

Its half =4|19682=4A. 0R. 31P.

Or the base and perpendicular may be reduced to perches, and the contents may thence be obtained, thus :

ch. l. perches.
Perp. 13.28=27.12
Half the perp. 13.56 } By prob. 4, sect. 1, part 2.

perches. ch. l.
1. Base 49.52=24.38
½ perp. 13.56

29712
24760
14856
4952

160)671.4912(4A. 0R. 31P.
31

perches.

2. Perp. 27.12
Half-base 24.76

16272
18984
10848
5424

671.4912=4A. 0R. 31P.

But square perches may be reduced to acres, &c. rather more commodiously by dividing by 40 and 4, than by 160; thus,

4|0)67|1

4)16. 31

A. 4. 0. 31

perches.

3. Base 49.52
Perp. 27.12

9904
4952
34664
9904

1342.9824

671.4912=4A. 0R. 31P.

The map may be readily drawn, having the distance from either end of the base to the perpendicular given; as may be evident from the figure.

PROBLEM VII.

The contents of a triangular piece of ground and the base given, to find the perpendicular.

Divide the contents in perches by half the base in perches, and the quotient will give you the perpendicular in perches, and so in chains.

EXAMPLE.

PL. 1. *fig.* 16.

Let *BC* be a ditch, whose length is 24*ch.* 40*l.*, by which it is required to lay out a triangular piece of ground, whose contents shall be 4A. 1R. 10P. Required the perpendicular.

ch. l. Perches.
Base 24.40=49.6
Half the base=24.8

A. R. P.
4 1 10
4

17
40
—— *Perches.*
24.8)690(27.82
1940
2040
560
64
——— *Perches. ch. l.*
Answer, perp. 27.82=13.45

This perpendicular being laid on any part of the base, and lines run from its extremity to the ends of the base, will lay out the triangle (by cor. to theo. 13, geom.) so that the perpendicular may be set on that part of the base which is most convenient and agreeable to the parties concerned.

PRACTICAL QUESTIONS.

Ex. 1. What is the area of a parallelogram whose length is 12.25 and its height 8.5 four-pole chains?
Ans. 10A. 1R. 26P.

Ex. 2. What is the area of a square whose side is 70.25 two-pole chains? Ans. 124A. 1R. 1P.

Ex. 3. What is the area of a rhombus whose side is 60 perches, and its height 45 perches? Ans. 16A. 3R. 20P.

Ex. 4. What is the area of a rhomboides whose less angle is 40° and the including sides 80 and 25 four-pole chains?

Ans. 128A. 2R. 9P.

Ex. 5. What is the area of a triangle whose base is 12 and its perpendicular height 6 two-pole chains?

Ans. 1A. 3R. 8P.

1 = 3 = 24

LEMMA.

PL. 8. *fig.* 9.

If from half the sum of the sides of any plane triangle ABC each particular side be taken, and if the half-sum and the three remainders be multiplied continually into each other, the square root of this product will be the area of the triangle.

Bisect any two of the angles, as A and B, with the lines AD, BD, meeting in D; draw the perpendiculars DE, DF, DG.

The triangle AFD is equiangular to AED; for the angle $FAD=EAD$ by construction, and $AFD=AED$, being each a right angle, and of consequence $ADF=ADE$; wherefore AD : DE : : AD : DF; and since AD bears the same proportion to DF that it does to DE, $DF=DE$, and the triangle $AFD=AED$. The same way $DE=DG$, and the triangle $DEB=DGB$, and $FD=DE=DG$; therefore D will be the centre of a circle that will pass through E, F, G.

In the same way, if A and C were bisected, the same point D would be had; therefore a line from D to C will bisect C, and thus the triangles DFC, DGC will be also equal.

Produce CA to H, till $AH=EB$ or GB; so will HC be equal to half the sum of the sides, viz. to $\frac{1}{2}AB+\frac{1}{2}AC+\frac{1}{2}BC$; for FC, FA, EB are severally equal to CG, AE, BG; and all these together are equal to the sum of the sides of the triangle; therefore $FC+FA+EB$ or CH are equal to half the sum of the sides.

$FC=CH-AB$, for $AF=AE$, and $HA=EB$; therefore $HF=AB$, and $AF=CH-BC$; for $CF=CG$, and $AH=GB$; therefore $BC=HA+FC$, and $AC=CH-AH$.

Continue DC till it meets a perpendicular drawn upon H in K; and from K draw the perpendicular KI, and join AK.

Because the angles AHK and AIK are two right ones, the angles HAI and K together are equal to two right; since the angles of the two triangles contain four right: in the same way $FDE+FAE=$ (two right angles $=$) $FAE+IAH$; let FAE be taken from both, then $FDE=IAH$, and of course $FAE=K$; the quadrilateral figures $AFDE$ and $KHAI$ are therefore similar, and have the sides about the equal angles proportional;

and it is plain the triangles CFD and CHK are also proportional: hence,

$$FD : HA :: FA : HK$$
$$FD : FC :: HK : HC.$$

Wherefore, by multiplying the extremes and means in both, it will be the square of $FD \times HK \times HC = FC \times FA \times HA \times HK$: let HK be taken from both, and multiply each side by CH; then the square of $CH \times$ by the square of $FD = FC \times FA \times HA \times CH$.

It is plain by the foregoing problem, that $\frac{1}{2}AB \times DE + \frac{1}{2}BC \times DG + \frac{1}{2}AC \times FD =$ the area of the triangle; or that half the sum of the sides, viz. $CH \times FD =$ the triangle; wherefore, the square of $CH \times$ by the square of $FD = FC \times FA \times HA \times CH$, that is, the half-sum multiplied continually into the differences between the half-sum and each side will be the square of the area of the triangle, and its root the area. Q. E. D.

Cor 1. If all the sides be equal, the rule will become $\sqrt{\frac{3}{2}a \times \frac{1}{2}a \times \frac{1}{2}a \times \frac{1}{2}a} = \frac{1}{4}aa\sqrt{3}$, for the equilateral triangle whose side is a.

Hence the following problem will be evident.

PROBLEM VIII.

The three sides of a plane triangle given, to find the area.

RULE.*

From half the sum of the three sides subtract each side severally; take the logarithms of half the sum and three remainders, and half their total will be the logarithm of the area: or, take the square root of the continued product of the half-sum and three remainders for the area.

EXAMPLES.

PL. 8. *fig.* 9.

1. In the triangle ABC are

Given $\left\{\begin{array}{l} AB = 10.64 \\ AC = 12.28 \\ CB = 9.00 \end{array}\right\}$ four-pole chains; required the area.

Sum 31.92

* The demonstration of this is plain from the foregoing lemma, and the nature of logarithms.

Half-sum	15.96	Log.	1.203033
Remainders	5.32	—	0.725912
	3.68	—	0.565848
	6.96	—	0.842609
			2)3.337402
Answer, sqr. ch.	46.63	log.	1.668701
or,	4.663 acres.		

Or, $15.96 \times 5.32 \times 3.68 \times 6.96 = 2174.71113216$; the square root of which is 46.63, for the area, as before.

2. What quantity of land is contained in a triangle, the three sides of which are 80, 120, and 160 perches respectively?
Ans. 29A. 7P.

3. What quantity of land is contained in a triangle, the three sides of which are 20, 30, and 40 four-pole chains?
Ans. 29A. 7.579P.

4. How many acres are in a triangle, whose sides are 49, 50.25, 25.69 four-pole chains? Ans. 61A. 1R. 39.68P.

PROBLEM IX.

Two sides of a plane triangle and their included angle given, to find the area.

RULE.*

To the log. sine of the given angle (or of its supplement to 180° if obtuse) add the logarithms of the containing sides; the sum less radius will be the logarithm of the double area.

Or, As radius
Is to the sine of any angle of a triangle,
So is the product of the sides including the angle:
To double the area of the triangle.

That is, $\frac{AB \times AC \times \text{Nat. S. of } \angle A}{2}$ (Pl. 5, fig. 17) = the area.

* Demonstration. This follows from rule 2, prob. 5, and from the nature of logarithms, because a triangle is half a parallelogram of the same base and height.

Or thus, PL. 11, *fig.* 3.

Let AH be perpendicular to AB and equal to AC, and HE, FCG parallel to AB; then making $AH (=AC)$ radius, $AF (=CD)$ will be the sine of CAD, and the parallelograms $ABEH$ (the product of the given sides) and $ABGF$ (the double area of the triangle), having the same base AB, are in proportion as their heights AH, AF; that is, as radius to the sine of the given angle; which proportion gives the operation as in the rule above.

EXAMPLES.

PL. 5. *fig.* 16.

Suppose two sides *AB*, *AC* of a triangular lot *ABC* form an angle of 30 degrees, and measure one 64 perches, and the other 40.5, what must the contents be?

Given angle	30°	sine	9.698970
Containing sides	64	log.	1.806180
	40.5	log.	1.607455
	2)1296	log.	3.112605

160)648(4A. 8P., Answer.

8

Or thus:

.500000 sine $\angle A$
64 *AB*

32.000000
40.5 *AC*

2)1296.0000000

160)648

4A. 8P.

2. Required the area of a triangle, two sides of which are 49.2 and 40.8 perches, and their contained angle 144½ degrees.
Ans. 3A. 2R. 22P.

3. What quantity of ground is enclosed in an equilateral triangle, each side of which is 100 perches, either angle being 60 degrees?
Ans. 27A. 10P.

PROBLEM X.

To find the area of a trapezoid, viz. a figure bounded by four right lines, two of which are parallel, but unequal.

RULE.*

Multiply the sum of the parallel sides by their perpendicular distance, and take half the product for the area.

* Demonstration. The trapezoid *ABCD* (pl. 14, fig. 8) is equivalent to the rectangle contained by its altitude and half the sum of the parallel

EXAMPLES.

1. Required the area of a trapezoid, of which the parallel sides are, respectively, 30 and 49 perches, and their perpendicular distance 61.6.

$$\left.\begin{array}{l} 30+49=79 \\ 61.6 \end{array}\right\} \text{Multiply.}$$

$$2)4866.4$$

Answer, 2433.2=15A. 33.2P.

PL. 9. *fig.* 10.

2. In the trapezoid *ABCD* the parallel sides are, *AD* 20 perches, *BC* 32, and their perpendicular distance *AB* 26; required the contents. Ans. 4A. 36P.

PROBLEM XI.

To find the contents of a trapezium.

RULE I.*

Multiply the diagonal, or line joining the remotest opposite angles, by the sum of the two perpendiculars falling from the other angles to that diagonal, and half the product will be the area.

sides *BC* and *AD*. For draw *CE* parallel to *AB* (prob. 8), bisect *ED* in *F*, and draw *FG* parallel to *AB*, meeting the production of *BC* in *G*. Because *BC* is equal to *AE*, *BC* and *AD* are together equal to *AE* and *AD*, or to twice *AE* with *ED*, or to twice *AE* and twice *EF*, that is, to twice *AF*; consequently, $AF=\frac{1}{2}(BC+AD)$. Wherefore, the rectangle contained by the altitude of the trapezoid and half the sum of its parallel sides is equivalent to the rhomboid *BF*: but the rhomboid *EG* is equivalent to the triangle *ECD* (theo. 12, cor. 2); add to each the rhomboid *BE*, and the rhomboid *BF* is equivalent to the trapezoid *ABCD*.

Note.—On this proposition is founded the method of offsets, which enters so largely into the practice of land surveying. In measuring a field of a very irregular shape, the principal points only are connected by straight lines, forming sides of the component triangles, and the distance of each remarkable flexure of the extreme boundary is taken from these rectilineal traces. The exterior border of the polygon is therefore considered as a collection of trapezoids, which are measured by multiplying the mean of each pair of offsets or perpendiculars into their base or intermediate distance, which is one of the other sides, because the parallel sides are perpendicular to it.

* Demonstration. For the trapezium *ABDC* = the triangles *ABC*+*ADC*$=\frac{AC\times Bb}{2}+\frac{AC\times Dd}{2}=\frac{Bb+Dd}{2}\times AC$. Q. E. D.

EXAMPLE.

PL. 7. *fig.* 3.

Let *ABCD* be a field in form of a trapezium, the diagonal *AC* 64.4 perches, the perpendicular *Bb* 13.6, and *Dd* 27.2; required the contents.

Diagonal=64.4
13.64+27.2=40.8 } Multiply.

2)2627.52

160)131376(8A. 33¾P., Answer.
1280

33¾ perches.

Note.—The method of multiplying together the half-sums of the opposite sides of a trapezium for the contents is erroneous, and the more so the more oblique its angles are.

To draw the map, set off *Ab* 28 perches, and *Ad* 34.4, and there make the perpendiculars to their proper lengths, and join their extremities to those of the diagonal.

Note.—When one of the diagonals and the four sides of a trapezium are given, it is divided into two triangles whose sides are given; the area of each triangle may be found (by prob 8), and their sum will give the area of the trapezium.

RULE II.*

If there be drawn two diagonals cutting each other, the product of the diagonals multiplied by the natural sine of the angle of intersection of the diagonals will be double the area. And this rule is common to a square, rhombus, rhomboides, &c., as well as to all other quadrilateral figures; that is, $\frac{AC \times BD \times \text{Nat. S.} \angle R}{2}$ = the area. Pl. 14, fig. 9. Or, as radius : $S.\angle R$: : $\frac{1}{2}AC \times BD$: the area.

Note.—Because the diagonals of a square and rhombus intersect at a right angle, whose sine is 1, therefore half the product of their diagonals is the area.

* Demonstration. Pl. 14, fig. 9. For the trapez. = the four $\triangle$s $ARB, BRC, CRD, DRA = (AR \times RB + BR \times RC + CR \times RD + DR \times RA) \times \frac{1}{2} S.\angle R = (\overline{AR+RC} \times \overline{BR+CR+RA} \times DR) \times \frac{1}{2} S.\angle R = \overline{AR+RC} \times \overline{BR+RD} \times \frac{1}{2} S.\angle R = AC \times BD \times \frac{1}{2} S.\angle R$. Q. E. D.

2. A gentleman, knowing that the area of a circle is greater than that of any other figure of equal perimeter, walls in a circular deer-park of 100 perches diameter, in which he makes an elliptical fish-pond 10 perches long by 5 wide. Required the length of his wall, contents of his park, and area of his pond.

Answer. The wall 314.16 perches, enclosing 49A. 14P., of which 39¼ perches, or ¼ of an acre nearly, is appropriated to the pond.

PROBLEM XIII.

The area of a circle given, to find its diameter.

RULE.

To the logarithm of the area add 0.104909, and half the sum will be the logarithm of the diameter. Or, divide the area by .7854, and the square root of the quotient will be the diameter.

EXAMPLE.

A horse in the midst of a meadow suppose
Made fast to a stake by a line from his nose:
How long must this line be that, feeding all round,
Permits him to graze just an acre of ground?

Area in perches 160, log. 2.204120
0.104909

2)2.309029

Diameter
2)14.2733 log. 1.154514

Answer, 7.13665 per. = 117*ft.* 9*in.*

PROBLEM XIV.

Allowance for roads.

It is customary to deduct 6 acres out of 106 for roads; the land before the deduction is made may be termed the *gross*, and that remaining after such deduction the *neat*.

RULE.

The gross divided } by 1.06 { quotes the neat.
The neat multiplied } { produces the gross.

EXAMPLES.

1. How much land must I enclose to have 850A. 2R. 20P. neat?

40|20
4| 2.5

Acres. A. R. P.
850.625 × 1.06 = 901.6625 = 901 2 26, the answer.

2. How much neat land is there in a tract of 901A. 2R. 26P. gross?

40|26
4| 2.65

Acres. A. R. P.
1.06)901.6625(850.625 = 850 2 20, the answer.
848

&c.

Note.—These two operations prove each other.

PROBLEM XV.

To find the area of a piece of ground, be it ever so irregular, by dividing it into triangles and trapezia.

Pl. 7. *fig.* 4.

We here admit the survey to be taken and protracted; by having, therefore, the map, and knowing the scale by which it was laid down, the contents may be thus obtained.

Dispose the given map into triangles by fine pencilled lines, such as are here represented in the scheme, and number the triangles with 1, 2, 3, 4, &c. Your map being thus prepared, rule a table with four columns, the first of which is for the number of the triangle, the second for the base of it, the third for the perpendicular, and the fourth for the contents in perches.

Then proceed to measure the base of number 1, from the scale of perches the map was laid down by, and place that in the second column of the table, under the word base; and from the angle opposite to the base open your compasses so as when one foot is in the angular point, the other, being moved backward and forward, may just touch the base line, and neither go the least above nor beneath it; that distance in the compasses, measured from the same scale, is the length of that perpendicular, which place in the third column under the word perpendicular.

If the perpendiculars of two triangles fall on one and the same base, it is unnecessary to put down the base twice, but insert the second perpendicular opposite to the number of the triangle in the table, and join it with the other perpendicular by a brace, as Nos. 1 and 2, 4 and 5, 6 and 7, 9 and 10, &c.

Proceed after this manner till you have measured all the triangles, and then, by prob. 6, find the contents in perches of each respective triangle, which severally place in the table opposite to the number of the triangle, in the fourth column, under the word contents.

But where two perpendiculars are joined together in the table by a brace, having both one and the same base, find the contents of each (being a trapezium) in perches, by prob. 11, which place opposite the middle of those perpendiculars, in the fourth column, under the word contents.

Having thus obtained the contents of each respective triangle and trapezium which the map contains, add them all together, and their sum will be the contents of the map in perches, which being divided by 160 gives the contents in acres. Thus, for

EXAMPLE.

No.	Base.	Perpend.	Contents.
1	24.8	17.0 }	412.92
2		16.3 }	
3	28.2	16.0	225.6
4	39.8	19.6 }	712.42
5		16.2 }	
6	49.4	29.0 }	1086.8
7		15.0 }	
8	38.7	6.7	129.64
9	40.0	17.0 }	600
10		13.0 }	
11	42.8	10.2 }	481.5
12		12.3 }	
13	26.2	17.9	234.49
14	24.0	11.6 }	259.2
15		10.0 }	
Contents in perches - - -			4142.57

This being divided by 160 will give 25A. 3R. 22P., the contents of the map.

Let your map be laid down by the largest scale your paper will admit, for then the bases and perpendiculars can be measured with greater accuracy than when laid down by a smaller scale, and if possible measure from scales divided diagonally.

If the bases and perpendiculars were measured by four-pole chains, the contents of every triangle and trapezium may be had as before in problems 6 and 11, and consequently the whole contents of the map.

If any part of your map has short or crooked bounds, as those represented in plate 7, fig. 5, then by the straight edge of a transparent horn draw a fine pencilled line, as AB, to balance the parts taken and left out, as also another BC: these parts, when small, may be balanced very nearly by the eye, or they may be more accurately balanced by method the third. Join the points A and C by a line, so will the contents of the triangle ABC be equal to that contained between the line AC and the crooked boundary from A to B, and to C: by this method the number of triangles will be greatly lessened, and the contents become more certain; for the fewer operations you have the less subject will you be to err, and if an error be committed the sooner it may be discovered.

The lines of the map should be drawn small and neat, as well as the bases, the compasses neatly pointed, and the scale accurately divided; without all which you may err greatly. The multiplications should be run over twice at least, as also the addition of the column of contents.

From what has been said it will be easy to survey a field by reducing it into triangles and measuring the bases and perpendiculars by the chain. To ascertain the contents only it is not material to know at what part of the base the perpendicular was taken; since it has been shown (in cor. to theo. 13 geom.) that triangles on the same base and between the same parallels are equal: but if you would draw a map from the bases and perpendiculars, it is evident that you must know at what part of the base the perpendicular was taken, in order to set it off in its due position; and hence the map is easily constructed.

PROBLEM XVI.

Pl. 8. *fig.* 5.

To determine the area of a piece of ground, having the map given, by reducing it to one triangle equal thereto, and thence finding its contents.

Let *ABCDEFGH* be a map of ground which you would reduce to one triangle equal thereto.

Produce any line of the map, as *AH*, both ways; lay the edge of a parallel ruler from *A* to *C*, having *B* above it; hold the other side of the ruler, or that next you, fast; open till the same edge touches *B*, and by it, with a protracting pin, mark the point *b* on the produced line; lay the edge of the ruler from *b* to *D*, having *C* above it, hold the other side fast, open till the same edge touches *C*, and by it mark the point *c* on the produced line. A line drawn from *c* to *D* will take in as much as it leaves out of the map.

Again, lay the edge of the ruler from *H* to *F*, having *G* above it; keep the other side fast, open till the same edge touches *G*, and by it mark the point *g* on the produced line; lay the edge of the ruler from *g* to *E*, having *F* above it, keep the other side fast, open till the same edge touches *F*, and by it mark the point *f* on the produced line. Lay the edge of the ruler from *f* to *D*, having *E* above it, keep the other side fast, open till the same edge touches *E*, and by it mark the point *e* on the produced line. A line drawn from *D* to *e* will take in as much as it leaves out. Thus have you the triangle *cDe*, equal to the irregular polygon *ABCDEFGH*.*

If, when the ruler's edge is applied to the points *A* and *C*, the point *B* falls under the ruler, hold that side next the said points fast, and draw back the other to any convenient distance; then hold this last side fast, and draw back the former edge to *B*, and by it mark *b* on the produced line; and thus a parallel may be drawn to any point under the ruler as well as if it were above it. It is best to keep the point of your protracting pin in the last point in the extended line till you lay the edge of the ruler from it to the next station, or you may mistake one point for another.

This may also be performed with a scale or ruler which has a thin-sloped edge, called a fiducial edge, and a fine-pointed pair of compasses. Thus,

Lay that edge on the points *A* and *C*; take the distance from the point *B* to the edge of the scale, so that it may only touch it, in the same manner as you take the perpendicular of a tri-

* The demonstration of this is evident from prob. 19, Geom., page 63 of this book.

angle; carry that distance down by the edge of the scale parallel to it to *b*, and there describe an arc on the point *b*, and if it just touches the ruler's edge the point *b* is in the true place of the extended line. Lay then the fiducial edge of the scale from *b* to *D*, and take a distance from *C* that will just touch the edge of the scale; carry that distance along the edge till the point which was in *C* cuts the produced line in *c*; keep that foot in *c* and describe an arc, and if it just touches the ruler's edge the point *c* is in the true place of the extended line. Draw a line from *c* to *D* and it will take in and leave out equally: in like manner the other side of the figure may be balanced by the line *eD*.

Let the point of your compasses be kept to the last point of the extended line till you lay your scale from it to the next station, to prevent mistakes from the number of points.

That the triangle *cDe* is equal to the right-lined figure *ABCDEFGH* will be evident from problems 18, 19, geom.; for thereby, if a line were drawn from *b* to *C*, it will give and take equally, and then the figure *bCDEFGH* will be equal to the map. Thus the figure is lessened by one side, and the next balance line will lessen it by two, and so on, and will give and take equally. In the same manner an equality will arise on the other side.

The area of the triangle is easily obtained, as before, and thus you have the area of the map.

It is best to extend one of the shortest lines of the polygon; because if a very long line be produced, the triangle will have one angle very obtuse, and consequently the other two very acute; in which case it will not be easy to determine exactly the length of the longest side, or the points where the balancing lines cut the extended one.

This method will be found very useful and ready in small enclosures, as well as very exact; it may be also used in large ones, but great care must be taken of the points on the extended line, which will be crowded, as well as of not missing a station.

PROBLEM XVII.

A map with its area being given, and its scale omitted to be either drawn or mentioned, to find the scale.

Cast up the map by any scale whatsoever, and it will be
As the area found
Is to the square of the scale by which you cast up,
So is the given area of the map
To the square of the scale by which it was laid down.
The square root of which will give the scale.

EXAMPLE.

A map whose area is 126A. 3R. 16P. being given, and the scale omitted to be either drawn or mentioned, to find the scale.

Suppose this map was cast up by a scale of 20 perches to an inch, and the contents thereby produced be 31A. 2R. 34P.

As the area found, 31A. 2R. 34P.=5074P.
Is to the square of the scale by which it was cast up, that is, to $20 \times 20 = 400$,
So is the given area of the map 126A. 3R. 16P.=20296P.
To the square of the scale by which it was laid down.
5074 : 400 :: 20296 : 1600, the square of the required scale.

Root.
1600(40
16

8) 00

Answer. The map was laid down by a scale of 40 perches to an inch.

PROBLEM XVIII.

How to find the true contents of a survey, though it be taken by a chain that is too long or too short.

Let the map be constructed, and its area found, as if the chain were of the true length. And it will be,

As the square of the true chain
Is to the contents of the map,
So is the square of the chain you surveyed by
To the true contents of the map.

EXAMPLE.

If a survey be taken with a chain which is 3 inches too long, or with one whose length is 42 feet 3 inches, and the map thereof be found to contain 920A. 2R. 20P.; required the true contents.

As the square of 42*ft.* 0*in.* = the square of 504 inches = 254016
Is to the contents of the map, 920A. 1R. 20P.=147260P.,
So is the square of 42*ft.* 3*in.* = the square of 507 inches = 257049
To the true contents.

P. P.
250416 : 147260 : : 257049 : 149019
A. R. P.
160)149019(931 1 19, Answer.

501

219

40)59(1R.

19P.

METHOD OF DETERMINING THE AREAS OF RIGHT-LINED FIGURES UNIVERSALLY, OR BY CALCULATION.

Definitions.

PL. 8. *fig.* 7.

1. MERIDIANS are north and south lines, which are supposed to pass through every station of the survey.

2. The difference of latitude, or the northing or southing of any stationary line, is the distance that one end of the line is north or south from the other end; or, it is the distance which is intercepted on the meridian, between the beginning of the stationary line and a perpendicular drawn from the other end to that meridian. Thus, if *NS* be a meridian line passing through the point *A* of the line *AB*, then is *Ab* the difference of latitude or southing of that line.

3. The departure of any stationary line is the nearest distance from one end of the line to a meridian passing through the other end. Thus *Bb* is the departure or easting of the line *AB*: but if *CB* be a meridian, and the measure of the stationary distance be taken from *B* to *A*, then is *BC* the difference of latitude, or northing, and *AC* the departure or westing of the line *BA*.

4. That meridian which passes through the first station is sometimes called the first meridian; and sometimes it is a meridian passing on the east or west side of the map, at the distance of the breadth thereof, from east to west, set off from the first station.

5. The meridian distance of any station is the distance

thereof from the first meridian, whether it be supposed to pass through the first station or on the east or west side of the map.

THEOREM I.

In every survey which is truly taken, the sum of the northings will be equal to that of the southings; and the sum of the eastings equal to that of the westings.

PL. 9. *fig.* 1.

Let *abcefgh* represent a plot or parcel of land. Let *a* be the first station, *b*, the second, *c* the third, &c. Let *NS* be a meridian line; then will all lines parallel thereto, which pass through the several stations, be meridians also; as *ao*, *bs*, *cd*, &c., and the lines *bo*, *cs*, *de*, &c., perpendicular to those, will be the east or west lines or departures.

The northings, $ei+go+hq=ao+bs+cd+fr$, the southings: for let the figure be completed; then it is plain that $go+hq+rk=ao+bs+cd$, and $ei-rk=fr$. If to the former part of this first equation $ei-rk$ be added, and fr to the latter, then $go+hq+ei=ao+bs+cd+fr$, that is, the sum of the northings is equal to that of the southings.

The eastings, $cs+qa=ob+de+if+rg+oh$, the westings. For $aq+yo\ (az)=de+if+rg+oh$, and $bo=cs-yo$. If to the former part of this first equation $cs-yo$ be added, and bo to the latter, then $cs+aq=ob+de+if+rg+oh$; that is, the sum of the eastings is equal to that of the westings. Q. E. D.

SCHOLIUM.

This theorem is of use to prove whether the field-work be truly taken or not; for if the sum of the northings be equal to that of the southings, and the sum of the eastings to that of the westings, the field-work is right, otherwise not.

Since the proof and certainty of a survey depend on this truth, it will be necessary to show how the difference of latitude and departure for any stationary line, whose course and distance are given, may be obtained by the table usually called the Traverse Table.*

* This table is calculated by the first case of right-angled plane trigonometry, taught in the fifth section of the first part of this book, where the hypothenuse and an acute angle are given, to find the legs.

In the right-angled triangle *ABC* (Pl. 8, fig. 7), given the distance or hypothenuse *AB* 91 chains, links, or perches, the course or one of the acute angles *ABC* 41°; it is required to find the legs, or the difference of latitude and departure.

To find the difference of latitude and departure by the Traverse Table.

This table is so contrived, that by finding therein the given course, and a distance not exceeding 120 miles, chains, perches, or feet, the difference of latitude and departure is had by inspection: the course is to be found at the top of the table when under 45 degrees, but at the bottom of the table when above 45 degrees. Each column signed with a course consists of two parts, one for the difference of latitude, marked Lat., the other for the departure, marked Dep., which names are both at the top and bottom of these columns. The distance is to be found in the column marked Dist., next the left-hand margin of the page.

EXAMPLE.

In the use of this table, a few observations only are necessary.

1. If a station consist of any number of even chains or perches (which are almost the only measures used in surveying), the latitude and departure are found at sight under the bearing or course, if less than 45 degrees, or over it if more, and in a line with the distance.

2. If a station consist of any number of chains and perches, and decimals of a chain or perch, under the distance 10, the lat. and dep. will be found as above, either over or under the bearing; the decimal point or separatrix being removed one figure to the left, which leaves a figure to the right to spare.

If the distance be any number of chains or perches, and the decimals of a chain or perch, the lat. and dep. must be taken

---

As radius	90°	10.000000
is to AB,	91	1.959041
So is the sine of B	41°	9.816943
to AC	59.70	1.775984
As radius	90°	10.000000
is to AB,	91	1.959041
So is the sine of A	49°	9.877780
to BC	68.68	1.836821

Hence AC is the departure and BC the difference of latitude which correspond to those in the table. In the same manner the difference of latitude and departure to every degree in the table is calculated, by which the practitioner can at any time prove the exactness of those in the table.

out at two or more operations, by taking out the lat. and dep. for the chains or perches in the first place; and then for the decimal parts.

To save the repeated trouble of additions, a judicious surveyor will always limit his stations to whole chains or perches and lengths, which can commonly be done at every station save the last.

1. In order to illustrate the foregoing observations, let us suppose a course or bearing to be *S.* 35° 15′ *E.*, and the distance 79 four-pole chains. Under 35° 15′, or 35¼ degrees, and opposite 79, we find 64.51 for the latitude, and 45.59 the departure, which signify that the end of that station differ in latitude from the beginning 64.51 chains, and in departure 45.59 chains.

Note.—We are to understand the same things if the distance is given in perches or any other measures, the method of proceeding being exactly the same in every case.

Again, let the bearing be 54¾ degrees, and distance as before; then over said degrees we find the same numbers, only with this difference, that the lat. before found will now be the dep., and the dep. the lat., because 54¾ is the complement of 35¼ degrees to 90, viz. lat. 45.59, dep. 64.51.

2. Suppose the same course, but the distance 7 chains 90 links, or as many perches. Here we find the same numbers, but the decimal point must be removed one figure to the left.

Thus, under 35¼, and in a line with 79 or 7.9, are

Lat. 6.45
Dep. 4.56

the 5 in the dep. being increased by 1, because the 9 is rejected; but over 54¾ we get

Lat. 4.56
Dep. 6.45

3. Let the course be as before, but the distance 7.79, then opposite

7.70	Lat. 6.29	Dep. 4.43
9	7	6
7.79	6.36	4.49

Or opposite

7.00	Lat. 5.72	Dep. 4.03
.79	.64	.46
7.79	6.36	4.49

THEOREM II.

When the first meridian passes through the map.

If the east meridian distances in the middle of each line be multiplied into the particular southing, and the west meridian distances into the particular northing, the sum of these products will be the area of the map.

PL. 10. *fig.* 1.

Let the figure *abkm* be a map, the lines *ab*, *bk* to the southward, and *km*, *ma* to the northward, NS the first meridian line passing through the first station *a*.

$$\left.\begin{matrix}\text{The meridian}\\ \text{distances east}\end{matrix}\right\} \left\{\begin{matrix} zd \times ao \\ tu \times ox\ (bq)\end{matrix}\right\} = \text{Area} \left\{\begin{matrix} am \\ ow\end{matrix}\right.$$

$$\left.\begin{matrix}\text{The meridian}\\ \text{distances west}\end{matrix}\right\} \left\{\begin{matrix} ef \times gx \\ hh \times ga\ (my)\end{matrix}\right\} = \text{Area} \left\{\begin{matrix} xp \\ gl\end{matrix}\right.$$

These four areas $am+ow+xp+gl$ will be the area of the whole figure *cmswiprlc*, which is equal to the area of the map *abkm*. Complete the figure.

The parallelograms *am* and *ow* are made of the east meridian distances *dz* and *tu* multiplied into the southings *ao* and *ox*; the parallelograms *xp* and *gl* are composed of the west meridian distances *ef* and *hh* multiplied into the northings *xg* and *ga* (*my*): but these four parallelograms are equal to the area of the map; for if from them be taken the four triangles marked *Z*, and in the place of those be substituted the four triangles marked *O*, which are equal to the former, then it is plain the area of the map will be equal to the four parallelograms. Q. E. D.

THEOREM III.

If the meridian distance when east be multiplied into the southings, and the meridian distance when west be multiplied into the northings, the sum of these less by the meridian distance when west multiplied into the southings is the area of the survey.

PL. 10. *fig.* 2.

Let *abc* be the map.

The figure being completed, the rectangle *af* is made of the meridian distance *eq* when east multiplied into the southing *an*; the rectangle *yk* is made of the meridian distance *xw*, multiplied into the northings *cz* or *ya*. These two rectangles, or parallelograms, $af+yk$, make the area of the figure *dfnyikd*; from which taking the rectangle *oy*, made of the meridian distance *tu* when west into the southings *oh* or *bm*, the remainder

is the area of the figure *dfohikd*, which is equal to the area of the map.

Let *bou*$=Y$, *urih*$=L$, *ric*$=O$, *wrc*$=Z$, *akw*$=K$, *efb*$=B$, and *ade*$=A$. I say that $Y+Z+B=K+L+A$.

$Y=L+O$; add Z to both, then $Y+Z=L+O+Z$: but $Z+O=K$, put K instead of $Z+O$, then $Y+Z=L+K$; add to both sides the equal triangles B and A, then $Y+Z+B=L+K+A$. If therefore $B+Y+Z$ be taken from *abc*, and in lieu thereof we put $L+K+A$, we shall have the figure *dfohikd* $=$*abc*; but that figure is made up of the meridian distance when east multiplied into the southing, and the meridian distance when west multiplied into the northing less by the meridian distance when west multiplied into the southing. Q. E. D.

COROLLARY.

Since the meridian distance when west multiplied into the southing is to be subtracted, by the same reasoning the meridian distance when east multiplied into the northing must be also subtracted.

SCHOLIUM.

From the two preceding theorems we learn how to find the area of the map when the first meridian passes through it; that is, when one part of the map lies on the east and the other on the west side of that meridian. Thus,

RULE.

The merid. dist. when $\left\{\begin{matrix}\text{east}\\\text{west}\end{matrix}\right\}$ multiplied into the $\left\{\begin{matrix}\text{southings,}\\\text{northings,}\end{matrix}\right\}$ their sum is the area of the map.

But,

The merid. dist. when $\left\{\begin{matrix}\text{east}\\\text{west}\end{matrix}\right\}$ multiplied into the $\left\{\begin{matrix}\text{northings,}\\\text{southings,}\end{matrix}\right\}$ the sum of these products taken from the former gives the area of the map.

These theorems are true when the surveyor keeps the land he surveys on his right-hand, which we suppose through the whole to be done; but if he goes the contrary way, call the southings northings and the northings southings, and the same rule will hold good.

General Rule for finding the Meridian Distances.

1. The meridian distance and departure both east or both west, their sum is the meridian distance of the same name.

2. The meridian distance and departure of different names, that is, one east and the other west, their difference is the meridian distance of the same name with the greater.

Thus, in the first method of finding the area, as in the following field-book,

The first departure is put opposite the northing or southing of the first station, and is the first meridian distance of the same name. Thus, if the first departure be east, the first meridian distance will be the same as the departure, and east also, and if west it will be the same way.

The first meridian distance	6.61 E.
The next departure	6.61 E.
The second meridian distance	13.22 E.
The next departure	1.80 E.
The third meridian distance	15.02 E.
At station 5, the meridian distance	5.78 E.
The next departure	7.76 W
The next meridian distance	1.98 W.
At station 11, the meridian distance	0.12 W.
The next departure	5.84 E.
The next meridian distance	5.72 E.

PL. 10. *fig.* 3.

In the 5th and 11th stations, the meridian distance being less than the departures and of a contrary name, the map will cross the first meridian, and will pass, as in the 5th line, from the east to the west line of the meridian; and in the 11th line it will again cross from the west to the east side, which will evidently appear if the field-work be protracted, and the meridian line passing through the first station be drawn through the map.

The field-book cast up by the first method will be evident

from the two foregoing theorems, and therefore requires no further explanation; but *to find the area by the second method* take this

RULE.

When the meridian distances are east, put the products of north and south areas in their proper columns, but when west in their contrary columns; that is, in the column of south area when the difference of latitude is north, and in north when south: the reason of which is plain from the last two theorems. The difference of these two columns will be the area of the map.

Construction of the Map from either the first or the second Table.

PL. 10. *fig.* 3.

Draw the line NS for a north and south line, which call the first meridian; in this line assume any point, as 1, for the first station. Set the northing of that stationary line, which is 3.54, from 1 to 2, on the said meridian line. Upon the point 2 raise a perpendicular to the eastward, the meridian distance being easterly, and upon it set 13.22, the second number in the column of meridian distances from 2 to 2, and draw the line 1, 2 for the first distance line: from 2 upon the first meridian set the northing of the second stationary line, that is, 9.65, to 3, and on the point 3 erect a perpendicular eastward, upon which set the meridian distance of the second station 16.82, from 3 to 3, and draw the line 2, 3, for the distance line of the second station. And since the third station has neither northing nor southing, set the meridian distance of it 33.02, from 3 to 4, for the distance line of the third station. To the fourth station there is 29.44 southing, which set from 3 to 5; upon the point 5 erect the perpendicular 5, 5; on which lay 13.54, and draw the line 4 to 5.

In the like manner proceed to set the northings and southings on the first meridian, and the meridian distances upon the perpendiculars raised to the east or west; the extremities of which connected by right lines will complete the map.

Field-book, Method I.

No. St.	Bearings.	C. L.	Lat. and half Dep.	Merid. Dist.	Area.	Deduct.
1	NE 75°	13.70	N 3.54 E 6.61	6.61 E 13.22 E		23.3994
2	NE 20½	10.30	N 9.67 E 1.80	15.02 E 16.82 E		144.9430
3	East.	16.20	0.00 E 8.10	24.92 E 33.02 E		
4	SW 33½	35.30	S 29.44 W 9.74	23.28 E 13.54 E	685.3632	
5	SW 76	16.00	S 3.87 W 7.76	5.78 E 1.98 W	22.3686	
6	North.	9.00	N 9.00 0.00	1.98 W 1.98 W	17.8200	
7	SW 84	11.60	S 1.21 W 5.77	7.75 W 13.52 W		9.3775
8	NW 53¼	11.60	N 6.94 W 4.64	18.16 W 22.80 W	126.0304	
9	NE 36¾	19.20	N 15.38 E 5.74	17.06 W 11.32 W	262.3828	
10	NE 22½	14.00	N 12.93 E 2.68	8.64 W 5.96 W	111.7152	
11	SE 76¾	12.00	S 2.75 E 5.84	0.12 W 5.72 E		0.3300
12	SW 15	10.85	S 10.48 W 1.40	4.32 E 2.92 E	45.2736	
13	SW 16¾	10.12	S 9.69 W 1.40	1.46 E 0.00	14.1474	
					1285.1012 178.0499	178.0499
			Contents in chains - - -		1107.0513	

The foregoing Field-book, Method II.

It is needless here to insert the columns of bearing or distances in chains, they being the same as before.

No. St.	Lat. and half Dep.	Merid. Dist.	N. Area.	S. Area.
1	N 3.54 E 6.61	6.61 E 13.22 E	23.3994	
2	N 9.65 E 1.80	15.02 E 16.82 E	144.9430	
3	0.00 E 8.10	24.92 E 33.02 E		
4	S 29.44 W 9.74	23.28 E 13.54 E		685.3632
5	S 3.87 W 7.76	5.78 E 1.98 W		22.3686
6	N 9.00 0.00	1.98 W 1.98 W		17.8200
7	S 1.21 W 5.77	7.75 W 13.52 W	9.3775	
8	N 6.94 W 4.64	18.16 W 22.80 W		126.0303
9	N 15.38 E 5.74	17.06 W 11.32 W		262.3828
10	N 12.93 E 2.68	8.64 W 5.96 W		111.7152
11	S 2.75 E 5.84	0.12 W 5.72 E	0.3300	
12	S 10.48 W 1.40	4.32 E 2.92 E		45.2736
13	S 9.69 W 1.46	1.46 E 0.00		14.1474
			178.0499	1284.1012
				178.0499
		Area in chains, as before,		1107.0513

A Specimen of the Pennsylvania Method of CALCULATION; which for its simplicity and ease in finding the Meridian Distances is supposed to be preferable in practice to any thing heretofore published on the subject.

Find, in the first place, by the Traverse Table, the lat. and dep. for the several courses and distances, as already taught; and if the survey be truly taken, the sums of the northings and southings will be equal, and also those of the eastings and westings. Then, in the next place, find the meridian distances, by choosing such a place in the column of eastings or westings as will admit of a continual addition of one, and subtraction of the other; by which means we avoid the inconvenience of changing the denomination of either of the departures.

The learner must not expect that in real practice the columns of lat. and those of dep. will exactly balance when they are at first added up, for little inaccuracies will arise, both from the observations taken in the field and in chaining; which to adjust, previous to finding the meridian distances, we may observe, that if in small surveys the difference amount to two-tenths of a perch for every station, there must have been some error committed in the field; and the best way in this case will be, to rectify it on the ground by a resurvey, or at least as much as will discover the error. But when the differences are not within those limits, the columns of northing, southing, easting, and westing may be corrected as follows:

Add all the distances into one sum, and say, as that sum is to each particular distance, so is the difference between the sums of the columns of northing and southing to the correction of northing or southing belonging to that distance: the corrections thus found are respectively additive when they belong to the column of northing or southing which is the less of the two, and subtractive when they belong to the greater; if the course be due east or west, the correction is always additive to the less of the two columns of northing or southing. The corrections of easting and westing are found exactly in the same manner.

The following example will sufficiently illustrate the manner of applying the rule.

EXAMPLE OF CORRECTING A SURVEY.

Field-Notes.			From the Tables.				Corrected.			
No.	Courses.	Dist. Per.	N.	S.	E.	W.	N.	S.	E.	W.
1	S 40° W	70		53.6		45.0		53.6		45.0
2	N 45° W	89	62.9			62.9	62.9			62.9
3	N 36 E	125	101.1		73.5		101.2		73.5	
4	North.	54	54.0				54.0			
5	S 81 E	186		29.1	183.7			29.0	188.6	
6	S 8 W	137		135.7		19.1		135.6		19.2
7	West.	130				130.0	00.1			130.0
			218.0	218.4	257.2	257.0	218.2	218.2	257.1	257.1
				218.0	257.0					
			Diff.	=.4	.2 = Diff.					

In this example, the sum of the distances is 791, and the difference between the columns of northing and southing is .4, also the first distance is 70; say then,

791 : 70 : : .4 : .04,

which fourth proportional .04 is the first correction belonging to the southing 53.6, from which the correction .04 should be subtracted.

In this manner the several corrections of the southings

$$\left.\begin{matrix}53.6\\29.1\\135.7\end{matrix}\right\} \text{are found to be} \left\{\begin{matrix}.04\\.09\\.07\end{matrix}\right\} \text{respectively.}$$

But as only two of these corrections amount to half a tenth, we must use .1 for each of the corrections .09 and .07, and neglect the correction .04; thus the correct southings become 53.6, 29.0, 135.6.

In like manner from the remaining distances we obtain to

$$\text{the northings} \left\{\begin{matrix}62.9\\101.1\\54.0\\00.0\end{matrix}\right\} \text{the additive corrections} \left\{\begin{matrix}.04\\.06\\.03\\.07\end{matrix}\right.$$

And consequently, by neglecting .04 and .03, and using .1 for each of the two .06 and .07, the northings when corrected are 62.9, 101.2, 54.0, 00.1.

In obtaining these corrections, it is commonly unnecessary to use all the significant figures of the distances: thus, for the ratio of 791 to 70, we may say, as 80 to 7.

The latitudes and departures being thus balanced, proceed to insert the meridian distances by the above method, where we still make use of the same field-notes, only changing chains and links into perches and tenths of a perch. Then by looking along the column of departure, it is easy to observe, that in the columns of eastings opposite station 9 all the eastings may be added, and the westings subtracted, without altering the denomination of either. Therefore, by placing 46.0, the east departure belonging to this station, in the column of meridian distances, and proceeding to add the eastings and subtract the westings, according to the rule already mentioned, we shall find that at station 8 these distances will end in 0, 0, or a cipher, if the additions and subtractions be rightly made. Then multiplying the upper meridian distance of each station by its respective northing or southing, the product will give the north or south area, as in the examples already insisted on, and which is fully exemplified in the annexed specimen. When these products are all made out and placed in their respective columns, their difference will give double the area of the plot, or twice the number of acres contained in the survey. Divide this remainder by 2, and the quotient thence arising by 160 (the number of perches in an acre), then will this last quotient exhibit the number of acres and perches contained in the whole survey; which in this example may be called 110 acres, 103 perches, or 110 acres, 2 roods, 23 perches.

FIELD-NOTES of the two foregoing methods, as practised in Pennsylvania.

Cast up by perches and tenths of a perch.

N.	Courses.	Dist.	N.	S.	E.	W.	M. D.	N. Area.	S. Area.
1	N 75° 00′ E	54.8	14.2		52.9		235.3 288.2	3341.26	
2	N 20.30 E	41.2	38.6		14.4		302.6 317.0	11680.36	
3	East.	64.8			64.8		381.8 446.6		
4	S 33.30 W	141.2		117.7		77.9	368.7 290.8		43395.99
5	S 76.00 W	64.0		15.5		62.1	228.7 166.6		3544.85
6	North.	36.0	36.0				166.6 166.6	5977.60	
7	S 84.00 W	46.4		4.9		46.1	120.5 74.4		590.45
8	N 53.15 W	46.4	27.8			37.2	37.2 00.0	1034.16	
9	N 36.45 E	76.8	61.5		46.0		46.0 92.0	2829.00	
10	N 22.30 E	56.0	51.7		21.4		113.4 134.8	5862.78	
11	S 76.45 E	48.0		11.0	46.7		181.5 228.2		1996.50
12	S 15.00 W	43.4		41.9		11.2	217.0 205.8		9092.30
13	S 16.45 W	40.5		38.8		11.7	194.1 182.4		7531.08
			229.8	229.8	246.2	246.2		30745.16	66151.17 30745.16
								2	35406.01
								Area in perches	177030.05

Note.—In the foregoing methods the first meridian passes through the map; but as it is more convenient to have it pass through the extreme east or west point of the same, I have given the following example to illustrate this method.

Of computing the area of a survey by having the bearings and distances given, geometrically considered and demonstrated.

Let *BCDEFGHA*, pl. 14, fig. 11, represent the boundary of a survey of which the following field-notes are given; it is required to find the area.

EXAMPLE.

Sides of the land.	Bearings.	Length in chains.
BC	East.	4.00
CD	N 9° E	4.00
DE	S 69 E	5.56
EF	S 36 E	7.00
FG	S 42 W	4.00
GH	S 75 W	10.00
HA	N 39 W	7.50
AB	N 42 E	5.00

RULE I.

Find the difference of latitude and departure answering to each course and distance by the Traverse Table or right-angled plane trigonometry, according to the directions already given, and place them under the succeeding columns North or South, East or West, according as they are north or south, east or west; then if the survey does not close, correct the errors by saying,* as the sum of all the distances is to each

* This arithmetical rule was given by Mr. Bowditch in his solution of Mr. Patterson's question of correcting a survey in No. 4 of the Analyst. Also, the editor, Dr. Adrain, has given precisely the same practical rule,

particular distance, so is the whole error in departure to the correction of the corresponding departure, each correction being so applied as to diminish the whole error in departure: pro-

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in his elegant solution of the said question, analytically demonstrated. As the demonstration of this important rule may give great satisfaction to those who have not an opportunity of seeing the Analyst, I have inserted Mr. Bowditch's demonstration of said rule, which is as follows, viz.

Demonstration 1. That the error ought to be apportioned among *all* the bearings and distances.

2. That in those lines in which an alteration of the measured distance would tend considerably to correct the error of the survey, a correction ought to be made; but when such an alteration would not have that tendency, the length of the line ought to remain unaltered.

3. In the same manner, an alteration ought to be made in the observed bearings, if it would tend considerably to correct the error of the survey, otherwise not.

4. In cases where alterations in the bearings and distances will both tend to correct the error it will be proper to alter them both, making greater or less alterations according to the greater or less efficacy in correcting the error of the survey.

5. The alterations made in the observed bearing and length of any one of the boundary lines ought to be such that the combined effect of such alterations may tend wholly to correct the error of the survey.

Suppose now that $ABCDE$ (pl. 14, fig. 12) represent the boundary lines of a field, as plotted from the observed bearings and lengths, and that the last point E, instead of falling on the first A, is distant from it by the length AE. The question will then be, what alterations BB', CC', DD''', &c. must be made in the positions of the points B, C, D, &c. so as to obtain the most probable boundaries $AB'C''D'''A$? If AB' be supposed to be the most probable bearing and length of the first boundary line, the point B would be moved through the line BB', and the following points C, D, E would in consequence thereof be moved in equal and parallel directions to C', D', E', and the boundary would become $AB'C'D'E'$. Again, if by correcting in the most probable manner the error in the observed bearing and length of BC (or $B'C'$), the point C' be moved to C'', the points D' and E' would be moved in equal and parallel directions to D'' and E'', and the boundary line would become $AB'C''D''E''$. In a similar manner, if by correcting the probable error in the bearing and length of CD (or $C''D''$) the point D'' be moved to D''', the point E'' would be moved in an equal and parallel direction to E''', and the boundary would become $AB'C''D'''E'''$. Lastly, by correcting the probable error in the bearing and length of the line DE (or $D'''E'''$) the true boundary $AB'C''D'''A$ would be obtained. If we suppose the lines $BB'CC''DD'''$, &c. to be parallel to AE, it would satisfy the second, third, fourth, and fifth of the preceding principles. For the change of position of the points B, C, &c. being in directions parallel to AE, the whole tendency of such change would be to move the point E directly towards A, conformably to the fifth principle; and by inspecting the figure, it will appear that the second, third, and fourth principles would also be satisfied. For, in the first place, it appears that the bearing of the first line AB would be altered considerably, but the length but little. This is agreeable to those principles, because an increase of the distance AB would move the point E in the direction Eb parallel to AB, and an alteration in the bearing would move it in the direction Eb' perpendicular to

ceed the same way for the corrections in latitudes. These corrections being applied to their corresponding differences of latitude and departure, that is, add when of the same name and

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AB. Now the former change would not tend effectually to decrease the distance *AE*, but the latter would be almost wholly exerted in producing that effect. Again, the length of the line *BC* would be considerably changed without altering essentially the bearing; the former alteration would tend greatly to decrease the distance *AE*, but the latter would not produce so sensible an effect. Similar remarks may be made on the changes in the other bearings and distances, but it does not appear to be necessary to enter more largely on this subject.

It remains now to determine the proportion of the lines *BB'*, *CC''*, *DD'''*, &c. To do this we shall observe, that in measuring the lengths of any lines the errors would probably be in proportion to their lengths. These supposed errors must, however, be decreased on those lines where the effect in correcting the error of the survey would be small, by the second and fourth principles.

In observing the bearings of all the boundary lines equal errors are liable to be committed; however, it will be proper, by the third and fourth principles to suppose the error greater or less in proportion to the greater or less effect it would produce in correcting the error of the survey.

Now the error of an observed bearing being given, as for example *GFI* (pl. 14, fig. 13), the change of position *GI* of the end of the line *G* would be proportional to the length of the line *FG* (=*FI*), so that the supposed errors both in the length and in the bearing of any boundary line would produce changes in the position of the end of it proportional to its length. There appears, therefore, a considerable degree of probability in supposing the lines *BB'*, *CC''*, *DD'''*, &c. to be respectively proportional to the lengths of the boundary lines *AB*, *BC*, *CD*, &c. The main point to be ascertained before adopting this hypothesis is, whether a due proportion of the error of the survey is thrown on the bearings and lengths of the sides. Now it is plain by this hypothesis that the error in any boundary line is supposed to be wholly in the bearing if the line be perpendicular to *AE*, and wholly in its length when parallel to *AE*; and if the length be the same in both cases, the change of position of the end of the line would in both cases be exactly equal. Thus, if *FGH* be the boundary line, *GI* the change of position of the point *B* in the former case, and *GH* in the latter, we should in this hypothesis have *GI*=*GH*.

To show the probability of this hypothesis it may be observed, that in measuring the lengths of a line *FGH* of six or eight chains of fifty links each, an error of one link might easily be committed by the stretching of the chain or the unevenness of the surface. This error would be about $\frac{1}{350}$ of the whole length. If we, therefore, suppose *GI* to be $\frac{1}{350}$ of *FG*, the angle *GFI* would be about 10'. Now, with such instruments as are generally made use of by surveyors, it is about as probable that an error of 10' was made in the bearing as that the above error, $\frac{1}{350}$ part, was made in measuring the length. We shall therefore adopt it as a principle, that the most probable way of apportioning the error of the survey *AE* is to take *BB'*, *CC''*, *DD'''*, &c. respectively proportional to the boundary lines *AB*, *BC*, *CD*, &c.

Hence the following practical rule for correcting a survey *geometrically*. Draw the boundary lines *ABCDE* by means of the observed bearings and

subtract when of different names, then the corrected difference of latitude and departure will be obtained, and the table will stand thus:

TABLE I.

Sids.	Courses.	D.Ch.	N.	S.	E.	W.	C.S.	C.W.	N.	S.	E.	W.
BC	East.	4.00			4.00		.02	.02		.02	3.98	
CD	N 9° E	4.00	3.95		0.63		.02	.02	3.93		0.61	
DE	S 69 E	5.56		1.99	5.19		.03	.02		2.02	5.17	
EF	S 36 E	7.00		5.66	4.11		.05	.03		5.71	4.08	
FG	S 42 W	4.00		2.97		2.68	.02	.02		2.99		2.70
GH	S 75 W	10.00		2.59		9.66	.06	.05		2.65		9.7
HA	N 39 W	7.50	5.82			4.72	.05	.04	5.77			4.7
AB	N 42 E	5.00	3.72		3.35		.03	.02	3.69		3.33	
			13.49	13.21	17.28	17.06	.28	.22	13.39	13.39	17.17	17.17
			13.21		17.06							
			.28 Er. S.		.22 Er. W.							

lengths, and find the error of the survey AE, and let the quotient of AE divided by the sum of all the lines AB, BC, CD, DE be represented by r; through the angular points B, C, D, &c. draw the lines BB', CC'', &c. parallel to AE, and in the same direction that A bears from E. Take $BB'=r\times AB$, $CC''=r\times(AB+BC)$, $DD'''=r\times(AB+BC+CD)$, &c.

The errors being corrected thus

As 47 : 4 : : .28 : .02
As 47 : 4 : : .28 : .02
&c. &c.
} The corrections of difference of lat. as in Table I.

As 47 : 4 : : .22 : .02
As 47 : 4 : : .22 : .02
&c. &c.
} The corrections of departure as in Table I.

The latitudes and departures being thus balanced, it is necessary to calculate the several meridian distances, in order to compute the area of the survey.

As beginning at the most easterly or most westerly point of the survey admits of a continual addition of the one and subtraction of the other, the most easterly or most westerly point can be easily discovered from the foregoing table, thus:

The first departure corrected is 3.98, which is the meridian distance of the second point of the survey from the first, to which add 0.61 the next dep. corrected, and their sum is 4.59, the meridian distance of the third points of the survey from the first; and in like manner 4.59+5.17=9.76= the meridian distance of the fourth point from the first, and 9.76+4.08=13.84 = the meridian distance east of the fifth point from the first; after the same manner, continue to add the dep. when east, but subtract when west: the next dep. is west, therefore 13.84 —2.70=11.14 = the meridian distance of the sixth point from the first, and 11.14 — 9.71=.43= the next. Now the next departure is 4.76, which is west, and 1.43 is the meridian distance of the seventh point from the first, which is east; therefore 4.76 — 1.43=3.33= the meridian distance of the eighth point from the first; as 3.33 is the greatest meridian distance west of the eighth point of the survey from the first, because the next departure is east 3.33; then, 3.33—3.33=0, which closes the survey: consequently, the eighth point of the survey is the most westerly point, and for the same reason as 13.84 is the greatest meridian distance east, which is the meridian distance of the fifth point of the survey. In like manner, the most easterly or

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Then through the points A, B', C'', D''', &c. draw the corrected boundary lines $ABCDA$, which being determined, the area may be found by dividing the figure into triangles in the usual method.

The proportional parts BB', CC'', &c. may be found expeditiously by means of a table of difference of latitude and departure, by finding the page where the sum of the lines $AB+BC+CD+DE$ in the distance column corresponds to AE in the departure or difference of latitude column; then find AB, $AB+BC$, &c. in the distance column, and the corresponding numbers will be equal to BB', CC'', DD''', &c. respectively.

most westerly point of the survey can be found by beginning at any other point.

After the most easterly or most westerly point of the survey is discovered, call that point the first station, and proceed to find the meridian distances for the several lines in the order in which they were surveyed; that is, the first dep. will be the first meridian distance, which place in the column of meridian distances opposite the said departure; to the same meridian distance add the said departure, to which sum add the next departure if it be of the same name with the foregoing departure, but subtract if it be of a different name, which sum or difference call the next meridian distance, and set it in the column of meridian distances opposite the departure last used; and in like manner, continue to add the departure twice when of the same name, but if of a different name subtract twice, and the last meridian distance will be zero, if the additions and subtractions are rightly performed; because the sum of the northings is equal to the sum of the southings after the survey is corrected, which is evident from theo. 1, and the foregoing table. Then,* multiplying the upper meridian distance of each station by the corresponding northing or southing, and place the product in the north or south area, according as the latitude is north or south, the difference of the sum of these products will give twice the area, half of which gives the area of the survey.

The most westerly point of the survey being made the first station, and the several meridian distances being calculated, &c., the foregoing table will stand thus:

* Demonstration. Let *NS* be a meridian passing through the most westerly station from the points *B*, *C*, *D*, *E*, *F*, *G*, and *H*; let fall the perpendiculars *Bb*, *Cb*, *Dd*, *Ec*, *Fe*, *Gf*, and *HI*, on the meridian *NS*:

Now, if from the area of the figure *dDEFGHI* the area of the figure *dDCBAHI* be taken, there remains the area of the survey. The area of the multangular figure *dDEFGHI* is equal to the sum of the areas of the trapezoids of which it is composed, viz. *dDEc*, *cEFe*, *eFGf*, and *fGHI*; but (by prob. 10), $(dD+cE)\times dc$ = twice the area of the trapezoid *dDEc*; and $dD+cE$ equal to the sum of the meridian distances of the points *D* and *E* from the first meridian line *NS*, and *dc* or *dg* = the southing of the point *E* from the point *D*. In like manner the area of every other trapezoid is found: but these are the south column areas: that is, $(dD+cE)\times dc+(cE+eF)\times ce+(eF+fG)\times ef+(fG+IH)\times fI$ = twice the area of the figure *dDEFGHI* = the sum of the south area column. And, in like manner, we demonstrate that $(dD+bC)\times db+bB\times bA+AI\times IH$ = twice the area of the figure *dDCBAHI* = the north area column; therefore, $(dD+cE)\times dc+(cE+eF)\times ce+(eF+fG)\times ef+(fG+IH)\times fI-[(dD+bC)\times db+bB\times bA+AI\times IH]$ = twice the area of the survey; consequently, the sum of the south area column — the sum of the north area column = twice the area of the survey. Q. E. D.

TABLE II.

St.	Courses.	D. C.	N.	S.	E.	W.	C. S.	C. W.	N.	S.	E.	W.	† M. D.	N. Area.	S. Area.
1*	N 42° E	5	3.72		3.35		.03	.02	3.69		3.33		3.33 6.66	12.2875	
2	East.	4			4.00		.02	.02		.02	3.98		10.64 14.62		.2128
3	N 9 E	4	3.95		0.63		.02	.02	3.93		0.61		15.23 15.84	59.8539	
4	S 69 E	5.56		1.99	5.19		.03	.02		2.02	5.17		21.01 26.18		42.4402
5	S 36 E	7		5 66	4.11		.05	.03		5.71	4.08		30.26 34.34		172.7846
6	S 42 W	4		2.97		2.68	.02	.02		2.99		2.70	31.64 28.94		94.6036
7	S 75 W	10		2.59		9.66	.06	.05		2.65		9.71	19.23 9.52		50.9595
8	N 39 W	7.50	5.82			4.72	.05	.04	5.77			4.76	4.76 0.00	27.4652	

9.6066 361.0007
99.6068
2)261.3939
Area 13A. 0R. 11P. 16)130.69695 S. C.
13,069695
4
0,278780
40
11,151200

RULE II.‡

The difference of latitude and departure being found and corrected as in the preceding rule.

* This is not the first station in the actual survey, but only the most westerly point of the survey as calculated by the foregoing method from the field-notes, which, for convenience' sake, I call the first station in making out this table.

† The meridian distances in this column are the sum of two adjacent meridian distances; but at the most westerly point the meridian distance is nothing, hence the first dep. is the first meridian distance, and, in like manner, the last dep. is the last meridian distance.

‡ Demonstration. Let us consider that every tract of land has an extreme southerly point, as *H*; and we reckon so much as any other point is distant from the east and west line *IK* (Pl. 14, fig. 11), that passes through

As beginning at the most northerly or most southerly point of the survey admits of a continual addition of the one and subtraction of the other, make choice of either of these points in order to calculate the area of the survey.

1. It is necessary to calculate the several latitudes in order to find the most northerly or most southerly point of the survey, which may be done from Table I., thus:

The first lat. is .02 south, which is the difference of latitude between the second point of the survey and the first, when the survey is corrected from the next departure 3.93, which is N., subtract .02 and their difference 3.91 is equal to the difference of latitude between the third point and the first, which is N., and 3.91—2.02=1.89 = the difference of lat. between the fourth point and the first; which is also N. But as the next difference of lat. is south, therefore 5.71—1.89=3.82 = the difference of lat. S. between the fifth point and the first; and 3.82+2.99=6.81= the difference of lat. S. between the sixth point and the first; and 6.81+2.65=9.46 = the difference of lat. S. between the seventh point and the first; and 9.46—5.77=3.69 = the difference of lat. S. between the eighth point and the first; and 3.69—3.69=0; hence it is evident that 9.46 is the greatest lat. S. = the difference of lat. between the seventh point and the first; therefore, the seventh point of the survey is the most southerly point; and, in like manner, 3.91 = the difference of lat. between the third point and the first, is the greatest lat. N.; hence, the third point is the most northerly point of the survey.

H, to be its latitude north, or the difference of latitude between the points *H* and *A*; *BL* the lat. of *B*; *CM* the lat of *C*; &c.

Thus, if from the contents of the figure *IABCDEFK*, the contents of the figure *FKIAHG* be subtracted, the remainder will be the area of the survey.

The multangular figure *IABCDEFK* is composed of all these trapezoids, viz. *IABL*, *BCML*, *CDNM*, *EOND*, and *FKOE*; but (by Prob. 10) $(IA+LB)\times IL$ = twice the area of the trapezoid *IABL*, and $(LB+CM)\times LM$ = twice the area of the trapezoid *BCML*, and so for the rest; and $IA+LB$ = the sum of the northings of the points *A* and *B* from the line *IK*, and IL = the easting of the point *B* from the point *A*. In like manner the area of every other trapezoid is found; but these are the east column areas, that is, $(IA+BL)\times IL+(BL+CM)\times LM+(CM+DN)\times MN+(DN+EO)\times NO+(EO+FK)\times OK$ = twice the area of the figure *IABCDEFK* = the sum of the east area column. And in like manner we demonstrate that $(FK+PG)\times PK$ = twice the area of the trapezoid *FKPG*; but $FK+PG$ = the lat. of *F*+ the lat. of *G* and PK = the dep. or westing of the point *G* from the point *F*, and $PG\times PH$ = twice the area of the triangle *PGH*, and $IA\times IH$ = twice the area of the triangle *IAH*; hence $(FK+PG)\times PK+PG\times PH+IA\times IH$ = twice the area of the figure *FKIAGH* = the sum of the west area column. Therefore $(IA+BL)\times IL+(BL+CM)\times LM+(CM+DN)\times MN+(DN+EO)\times NO+(EO+FK)\times OK-[(FK+PG)\times PK+PG\times PH+IA\times IH]$ = twice the area of the survey; consequently, the sum of the east area column — the sum of the west area column = twice the area of the survey. Q. E. D.

Now, by calling the most southerly point of the survey the first station, and proceeding to find the latitudes for the several lines in the order in which they were surveyed; that is, the first difference of lat. will be the first lat., which place in the column of latitudes, opposite the said difference of latitude; to the same lat. add the said difference of lat., to which sum add the next difference of lat. if it be of the same name, but subtract if of a different name, and place it in the column of latitudes; in like manner continue to add or subtract the difference of lat. twice, and the last lat. comes out nothing, if the additions and subtractions are rightly performed. Multiply each of the upper numbers in the column of latitudes by the corresponding dep., and place the products in the column of east or west area, according as the dep. is E. or W. The difference of these columns will be equal to twice the area, half of which will give the area of the survey; as in the following table.

TABLE III.

St.	Courses.	D. C.	N.	S.	E.	W.	c. s.	c. w.	N.	S.	E.	W.	Lat.	E. Area.	W. Area.
1	N 39° W	7.50	5.82			4.72	.05	.04	5.77				5.77 11.54		27.4662
2	N 42 E	5	3.72		3.35		.03	.02	3.69				15.23 18.92	50.7159	
3	East.	4			4.00		.02	.02			3.96	.02	18.90 18.88	75.2220	
4	N 9 E	4	3.95		0.63		.02	.02	3.93		0.61		22.81 26.74	13.9141	
5	S 69 E	5.56		1.99	5.19		.03	.02		2.02	5.17		24.72 22.70	127.8024	
6	S 36 E	7		5.66	4.11		.05	.03		5.71	4.06		16.99 11.28	69.3192	
7	S 42 W	4		2.97		2.66	.03	.02		2.99		2.70	8.29 5.30		23.3530
8	N 75 W	10	2.59			9.66	.06	.05		2.65		9.71	2.65 0.00		26.7315

336.9736 75.5797
75.5797
2)261.3939
10)130.69695
13.069695 Acres.

Ans. 13A. 0R. 11P.

Each of the numbers in the column of latitudes is twice the mean latitude of two adjacent latitudes; but at the most southerly point the latitude is nothing; hence the first difference of latitude is the first lat., and in like manner the last difference of lat. is the last latitude. It is also to be remarked that the first station used in this table is not the first station in the actual survey, but the most southerly point of the survey, as calculated by the foregoing method from Table I.

SECTION IV.

OF OFFSETS.

In taking surveys it is unnecessary and unusual to make a station at every angular point, because the field-work can be taken with much greater expedition by using offsets and intersections, and with equal certainty; especially where creeks, &c. bound the survey.

Offsets are perpendicular lines drawn or measured from the angular points of the land, that lie on the right or left-hand to the stationary distance, thus:

Pl. 11. *fig.* 2.

Let the black lines represent the boundaries of a farm or township; and let 1 be the first station: then if you have a good view to 2, omit the angular points between 1 and 2, and take the bearing and length of the stationary line 1, 2, and insert them in your field-book; but in chaining from 1 to 2, stop at *d* opposite the angular point *a*, and in your field-book insert the distance from 1 to *d*, which admit to be 4*ch.* 25*l.*, as well as the measure of the offset *ad*, which admit to be 1*ch.* 12*l.*, thus: by the side of your field-book, in a line with the first station, say at 4*ch.* 25*l.* L. 1*ch.* 12*l.*, that is, at 4*ch.* 25*l.* there is an offset to the left-hand of 1*ch.* 12*l.*

This done, proceed on your distance line to *e* opposite to the angle *b*, and measure *eb*; supposing then 1*e* to be 7*ch.* 40*l.*, and *eb* 3*ch.* 40*l.*, say (still in a line with the first station in your field-book) at 7*ch.* 40*l.* L. 3*ch.* 40*l.*, that is, at 7*ch.* 40*l.* there is an offset to the left of 3*ch.* 40*l.*; proceed then with your distance line to *f* opposite to the angle *c*, and measure *fc*; suppose then 1*f* to be 13*ch.* and *fc* 1*ch.* 25*l.*, say, in the same line as before, at 13*ch.* L. 1*ch.* 25*l.* Then proceed from *f* to 2, and you will have the measure of the entire stationary line 1, 2, which insert in its proper column by the bearing.

In taking offsets, it is necessary to have a perch chain, or a staff of half a perch, divided into links for measuring them; for by this means the chain in the stationary line is undisturbed, and the number of chains and links in that line from whence, or to which, the offsets are taken, may be readily known.

Having arrived at the second station, if you find your view will carry you to 3, take the bearing from 2 to 3, and in measuring the distance line, stop at *l* opposite *g*; admit 2*l* to be 4*ch.* 10*l.*, and the offset *lg* 1*ch.* 20*l.*, then in a line with the second station in your field-book, say at 4*ch.* 10*l.* R. 1*ch.* 20*l.*, that is, the offset is a right-hand one of 1*ch.* 20*l.* Again, at *m*, which suppose to be 10*ch.* 25*l.* from 2, take the offset *mh* of 1*ch.* 15*l.*, and in a line with the second station, say at 10*ch.* 25*l.* R. 1*ch.* 15*l.* In the same line, when you come to the boundary at *i*, insert the distance 2*i*, 13*ch.* 10*l.*, thus, at 13*ch.* 10*l.* 0; that is, at 13*ch.* 10*l.* there is no offset. At *n*, which is 15*ch.* from 2, take the offset *nk* 45*l.*, and still opposite to the second station say at 15*ch.* L. 45*l.*

Let the line 3, 6 represent the boundary which by means of water, briers, or any other impediment, cannot be measured. In this case make one or more stations within or without the land, where the distances may be measured, and draw a line from the beginning of the first to the end of the last distance, thus: make stations at 3, 4, and 5, take the bearings, and measuring the distances as usual, which insert in your field-book, and draw a mark like one side of a parenthesis, from the third to the fifth station, to show that a line drawn from the third station to the farthest end of the fifth stationary line will express the boundary. Thus,

No. Sta.	Deg.	*ch. l.*
3	172½	5.45
4	200	13.25
5	250	3.36

Suppose the point *p* of the boundary to be inaccessible by means of the lines 6*p* or *p*7 being overflowed, or that a quarry, furze, &c. might prevent your taking their lengths: in this case take the bearing of the line 6, 7, which insert opposite to the sixth station in your field-book with the other bearing; then direct the index to the point *p*, and insert its bearings on the left side of the field-book, opposite to the sixth station, annexing thereto the words *Int. for boundary;* and having measured and inserted the distance 6, 7, set the index in the direction of the line 7*p*, and insert its bearing on the left of the seventh station

of the field-book, annexing thereto the words *Int. for boundary*: the crossing or intersection of these two bearings will determine the point *p*, and of course the boundary 6*p*7 is also determined.

If your view will then reach in the first station, take its bearing, stationary line, and offsets as before, and you have the field-book completed. Thus,

The Field-book.

Remarks and Inter.	N. St.	Deg.	*ch. l.*	OFFSETS.
318 Int. to a tower	1	358	22.12	At 4*ch.* 25*l.* L. 1*ch.* 12*l.* at 7*ch.* 40*l.* L. 3*ch.* 40*l.* at 13*ch.* L. 1*ch.* 25*l.*
231½ Int. to ditto	2	297¾	22.12	At 4*ch.* 10*l.* R. 1*ch.* 20*l.* at 10*ch.* 25*l.* R. 1*ch.* 51*l.* at 13*ch.* 10*l.* 0. at 15*ch.* L. 45*l.*
	3	172¼	5.45	
	4	200	13.25	
	5	250	3.36	
155½ Int. for bound.	6	125	15.15	At 1*ch.* 20*l.* L. 2*ch.* 20*l.*
274 Int. for ditto.	7	105¼	15.10	at 7*ch.* 45*l.* L. 2*ch.* 32*l.* at 11*ch.* 25*l.* 0. at 12*ch.* 25*l.* R. 36*l.*

Close at the first station.

If you would lay down a tower, house, or any other remarkable object in its proper place, from any two stations take bearings to the object, and their intersection will determine the place where you are to insert it, in the manner that the tower is set out in the figure, from the intersection taken at the first and second stations of the above field-book.

A protraction of this will render all plain, on which lay off all your offsets and intersections, and proceed to find the contents by any of the methods in section the fourth.

The foregoing Field-book may be otherwise kept, thus:

Remarks and Intersection.	No. St.	Deg.	L. hand Offset. *ch. l.*	Dist. *ch. l.*	R. hand Offset. *ch. l.*
318 Int. to a tower -	1	358	1.12 3.40 1.25	4.25 7.40 13.00 22.12	
232½ Int. for ditto - -	2	297¾	 0.45	4.10 10.25 13.10 15.00 21.21	1.20 1.15
	3	172½		5.45	
	4	200		13.25	
	5	250		3.36	
155½ Int. for boundary	6	125		15.15	
274 Int. for boundary	7	105	2.20 2.32	1.20 7.45 11.25 12.25 15.10	 0.36

How to cast up offsets by the pen.

Pl. 11. *fig.* 2.

1, 2—1f=2f, 2f—1e=fe, 1e—1d=ed.

Then $1d \times \frac{1}{2}da = 1da$, and $ed \times \frac{1}{2}(da+eb) = beda$, $\frac{1}{2}(eb+fc) \times fe = befc$, and $2f \times \frac{1}{2}fc = cf9$; the sum of all which will be 1*abc*21; the area contained between the stationary line 1, 2 and the boundary 1*abc*2.

In the same manner you may find the area of 2*ihg*2, of *ik*3*i*, as well as what is without and withinside of the stationary line 7, 1.

If therefore the left-hand offsets exceed the right-hand ones, it is plain the excess must be added to the area within the stationary lines; but if the right-hand offsets exceed the left-hand ones the difference must be deducted from the said area, if the

ground be kept on the right-hand, as we have all along supposed; or in words thus:

To find the contents of offsets.

1. From the distance line take the distance to the preceding offset, and from that the distance of the one preceding it, &c. in four-pole chains; so will you have the respective distances from offset to offset, but in a retrograde order.

2. Multiply the last of these remainders by half the first offset, the next by half the sum of the first and second, the next by half the sum of the second and third, the next by half the sum of the third and fourth, &c. The sum of these will be the area produced by the offsets.

Thus, in the foregoing field-book the first stationary line is 22*ch.* 12*l.*, or 11*ch.* 12*l.* of four-pole chains. See the figure.

	ch. l.	*ch. l.*	*ch. l.*
From	11.12=1,	6.50=1*f*	3.90=1*e*
Take	6.50=1*f*	3.90=1*e*	2.25=1*d*
	4.62=2*f*	2.60=*ef*	1.65=*ed*

ch. l.

1*d*=2.25×32*l.*, half the first offset,	.7200
ed=1.65×1*ch.* 26*l.*, half the sum of the 1st and 2d, =	2.0790
ef=2.60×1*ch.* 32*l.*, half the sum of the 2d and 3d, =	3.4320
2*f*=4.62×37*l.*, half the last offset, =	1.7094
Contents of left offsets on the first distance in square four-pole chains,	7.9404

In like manner the rest are performed.

The sum of the left-hand offsets will be	14.0856
And the sum of the right-hand ones	3.6825
Excess of left-hand offets in sq. four-pole chains,	10.4031
Acres	1.04031
	.16124
	4
Perches	6.4496

Excess of left-hand offsets above the right-hand ones, 1A. 0R. 6P., to be added to the area within the stationary lines.

SECTION V.

To find the area of a piece of ground by intersections only, when all the angles of the field can be seen from any two stations on the outside of the ground.

PL. 12. *fig.* 1.

Let *ABCDEFG* be a field, *H* and *I* two places on the outside of it from whence an object at every angle of the field may be seen.

Take the bearing and distance between *H* and *I*; set that at the head of your field-book, as in the annexed one. Fix your instrument at *H*, from whence take the bearings of the several angular points *ABCD*, &c. as they are here represented by the lines *HA*, *HB*, *HC*, *HD*, &c. Again, fix your instrument at *I* and take bearings to the same angular points, represented by the lines *IA*, *IB*, *IC*, *ID*, &c., and let the first bearings be entered in the second column and the second bearings in the third column of your field-book; then it is plain that the points of intersection made from the bearings in the second and third columns of every line will be the angular points of the field, or the points *A*, *B*, *C*, *D*, &c., which points being joined by right lines will give the plan *ABCDEFGA* required.

Bear. 180, Dist. 28*ch.* of the Sta. *H* and *I*.

No.	Bear.	Bear.
A	261½	331½
B	265¾	317¼
C	248	307½
D	238¼	289
E	215½	262½
F	208½	286½
G	220	300

The same may be done from any two stations withinside of the land from whence all the angles of the field can be seen.

This method will be found useful in case the stationary distances from any cause prove inaccessible, or should it be required to be done by one party when the other, in whose possession it is, refuses to admit you to go on the land.

To find the contents of a field by calculation, which was taken by intersection.

In the triangle *AIH*, the angles *AHI*, *AIH*, and the base *HI* being known, the perpendicular *Aa* and the segments of the base *Ha*, *AI* may be obtained by trigonometry: and in the same manner all the other perpendiculars, *Bb*, *Cc*, *Dd*, *Ee*, *Ff*, *Gg*, and the several segments at *b*, *c*, *d*, *e*, *f*, *g*: if, therefore, the several perpendiculars be supposed to be drawn into the scheme (which are here omitted, to prevent confusion arising from a multiplicity of lines), it is plain that if from *bBCDEeb* there be taken *bBAGFeb* the remainder will be the map *ABCDEFGA*.

As before, half the sum of *Bb* and *Cc* multiplied by *bc* will be the area of the trapezium *bBCc*; after the same manner, half the sum of *Cc* and *Dd* multiplied by *cd* will give the area of the trapezium *cCDd*; and again, half the sum of *Dd* and *Ee* multiplied by *de* gives the area of the trapezium *dDEe*; and the sum of these three trapezia will be the area of the figure *bBCDEeb*.

Again, in the same manner, half the sum of *Bb* and *Aa* multiplied by *ab* will give the area of the trapezium *BbAa*, and half the sum of *aA* and *gG* by *ag* gives the trapezium *aAGg*; to these add the trapezia *gGFf* and *fFEe*, which are found in the like manner, and you will have the figure *bBAGFEeb*, and this taken from *bBCDEeb* will leave the map *ABCDEFGA*. Q. E. I.

It will be sufficient to protract this kind of work, and from the map to determine the area as well as in plate 10, fig. 3, to find the areas of the pieces 3, 4, 5, 6, 3 and 6, 7, 7, 6 from geometrical constructions.

How to determine the station where a fault has been committed in a field-book, without the trouble of going round the whole ground a second time.

From every fourth or fifth station, if they be not very long ones, or oftener if they are, let an intersection be taken to any object, as to any particular part of a castle, house, or cock of hay, &c., or, if all these be wanting, to a long staff with a white sheet or napkin set thereon, to render the object more conspicuous, and let this be placed on the summit of the land, and let the respective intersections so taken be inserted on the left-hand side of the field-book opposite to the stations from whence they were respectively taken.

In your protraction as you proceed let every intersection be laid off from the respective stations from whence they were taken, and let these lines be continued; if they all converge or

meet in one point, we thence conclude all is right, or so far as they do converge; but if we find a line of intersection to diverge or fly off from the rest, we may be sure that either a mistake has happened between the station the foregoing intersection was taken at and the station from whence the intersection line diverges, or there must be an error in the intersection; but to be assured in which of these the fault is, protract on to the next intersection, and having set it off, if it converges with the rest, though the foregoing one did not, we may conclude the fault was committed in taking the last intersection but one, and none in any station, and that so far is true as is protracted; but if this as well as the foregoing intersection diverge or fly from the point of concourse or converging point of the rest, the error must have its rise from some station or stations at or after that from whence the last converging intersection line was taken: so that by going to that station on the ground, and proceeding on to that where the next or from whence the following diverging intersection was taken, we can readily and with little trouble set all to rights.

But in most tracts of land one object cannot be seen from every station, or from perhaps one-fourth of them; in this case we are under the necessity to move the pole after we begin to lose sight of it, to some other part of the land, where it may be seen from as many more stations as possible; which is easily done by viewing the boundary before it be surveyed: the pole then being fixed in an advantageous place, the first intersection to it is best to be made from the same station from whence the last one was taken, and then as often as may be thought convenient, as before; in like manner the whole may be done by the removal of the pole.

When we here speak of stations, we do not mean such as are usually taken at every particular angle of the field: for it is to be apprehended, that every skilful surveyor, particularly such who use calculation, will take the longest distances possible, not only to lessen the number of stations, for the ease of either protraction or calculation, but with greater certainty to account for the land passed by, on the right-hand or on the left, which is taken by offsets: and surely it will be allowed that any measure taken on the ground, and the contents thence arithmetically computed, will be much more accurate than that which is obtained from any geometrical projection.

From what has been said it is plain, that from this method any fault committed in a survey can be readily determined, and therefore must be much preferable to the present method of taking diagonals, or the bearings and lengths of lines across

land, to accomplish that end; which last method is too frequently used by surveyors to approximate or arrive near the contents, which will ever remain uncertain, let these diagonals be ever so many, till the station or stations wherein the error or errors were committed be found; and the fault or faults be corrected.

Where one diagonal is taken, it may perhaps close or meet with one part of the survey and not with the other; in this case, if the surveyor would discover his error, he must survey that part of the land which did not close, and this may be half or more of the whole. And should the diagonal close with neither part, but be too long or too short, or should it fall on either side of the assigned point it was to close with, he ought to go over the whole, and make a new survey of it, in order to discover his error.

A number of diagonals are frequently taken, the sum of the lengths of which very often exceeds the circuit of the ground, and after all they are but approximations, and the contents remain uncertain as before; therefore, he who returns a map made up by the assistance of diagonals, where there remains a misclosure in any one part, runs the risk of being detected in an error, and must suffer uneasiness in his mind, as he cannot be certain of the return he makes.

The frequent misclosures which are botched up by diagonals occasion the many and frequent scandalous broils and animosities between surveyors, which tend to the loss of character of the one or the other, and indeed often to the disrepute of both, as well as to that of the science they profess.

But these may be easily remedied by intersections and the bearing or line to be adjusted where the fault was committed; and till this be found, nothing can be certain.

SECTION VI.

TO ENLARGE OR DIMINISH MAPS.

To enlarge or diminish a map, or to reduce a map from one scale to another; also, the manner of uniting separate maps of lands which join each other into one map of any assigned size.

Lay the map you would enlarge over the paper on which you would enlarge it, and with a fine protracting pin prick through every angular point of your map; join these points on your paper (laying the map you copy before you) by pencilled

or popped lines, and you have the copy of the map you are to enlarge: in this manner any protraction may be copied on paper, vellum, or parchment, for a fair map.

If you would enlarge a map to a scale which is double, or treble, or quadruple to that of the map to be enlarged, the paper you must provide for its enlargement must be two, or three, or four times as long and broad as the map; for which purpose in large things you will find it necessary to join several sheets of paper, and to cement them with white wafer or paste, but the former is best.

Then pitch upon any point in your copied map for a centre; from whence if distances be taken to its extreme points, and thence if those distances be set in a right line with (but from) the centre, and these last points fall within your paper, the map may be increased on it to a scale as large again as its own; and if the like distances be again set outwards in right lines from the centre, and if these last points fall within your paper, it will contain a map increased to a scale three times as large as its own, &c.

PL. 12. *fig.* 2.

Let the pricked or popped lines represent the copy of a down or old survey, laid down by a scale of 80 perches to an inch, and let it be required to enlarge it to one laid down by 40 to an inch.

Pitch upon your centre as ⊙, from whence through *a* lay the fiducial edge of a thin ruler; with a fine-pointed pair of compasses take the distance from *a* to the centre ⊙, and lay it by the ruler's edge from *a* to *A*; in the like manner take the distance from the next station *b* to the centre ⊙, and lay it over in a right line from *b* to *B*, and join the points *A* and *B* by the right line *AB*; in the like manner set over the distance from every station to the centre, from that station outwards, and you will have every point to enlarge to: the joining of these constantly as you go on by right lines will give you the enlarged map required.

In taking the distance from every station to the centre, set one foot of the compasses in the station, and the other very lightly over the centre-point, so lightly as scarcely to touch it, otherwise the centre-point will become so wide, that it may occasion several errors in the enlarged map: for if you err from the exact centre but a little, that error will become double, or treble, or quadruple, as you enlarge to a scale that is double, or treble, or quadruple of the given one; therefore great accuracy is required in enlarging a map.

When you have done with a station, give a dash with a pen or pencil to it, such as at the station *a* and *b*: by this means you cannot be disappointed in missing a station, or in laying your ruler over one station twice.

From what has been said it is plain, that if a map is to be enlarged to one whose scale is double the given one, that the distances from the respective stations to the centre, being set over by the ruler's edge, will give the points for the enlarged one. And thus may a map be enlarged from a scale of 160 to one of 80, from one of 80 to one of 40, from one of 20 to one of 10 perches to an inch, &c. For to enlarge to a scale that is double, the number of perches to an inch for the enlarged map must be half of those to an inch for that to be enlarged: to enlarge to a scale that is treble the given one, the number of perches to an inch for the enlarged map will be one-third of those for the other; if to a scale that is quadruple the given one, the number of perches to an inch for the enlarged map will be one-fourth of those for the other, &c.: therefore, if you would enlarge a map which is laid down by a scale of 120 perches to an inch to one of 40 perches to an inch, the distance from the several stations to the centre, being set twice beyond the said stations, will mark out the several points required, for these points will be three times farther from the centre than the stationary points of the map are.

In the same manner, if you would enlarge a map from a scale of 160 to one of 40 perches to an inch, the distance from the several stations to the centre, being set three times beyond said stations, will lay out the points for your enlarged map, for these points will be four times farther from the centre than are the stations of the map.

When a map is enlarged to another, whose scale is double, or treble, or quadruple, &c. of the given one, every line, as well as the length and breadth of the enlarged map, will be double, or treble, or quadruple, &c. those of the given one, for it must be easy to conceive that those maps are like; but the area if the scale be double will be four times, if treble nine times, if quadruple sixteen times that of the given figure; that is, it will contain four, nine, or sixteen times as many square inches as the given one (for it has been shown that like polygons are in a duplicate proportion with the homologous sides). Yet these figures being cast up by their respective scales will produce the same contents.

Thus much is sufficient for enlarging maps, and from hence, diminishing of them will be obvious; for one-fourth, one-third, or half the distances from the several stations to the centre

will mark out points which if joined will compose a map similar to the given one, whose scale will be four times, three times, or twice as small as the given one.

Thus, if we would reduce a map of from 40 to 80, from 20 to 40, from 10 to 20 perches to an inch, &c., half the distance of the stations from the centre will give the points requisite for drawing the map; if we would reduce from 40 to 120, from 20 to 60, from 10 to 30 perches to an inch, &c., one-third of the distances to the centre will give the points for the map; and if we would reduce from 40 to 160, from 20 to 80, from 10 to 40 perches to an inch, &c., one-fourth of the distances to the centre will give the points for the map.

By the methods here laid down I have reduced a map from a scale of 40 to one of 20 perches to an inch, which contained upwards of 1200 acres, and consisted of 224 separate divisions, without the least confusion from the lines; for none can arise if the methods here laid down be strictly observed.

I have also from the same methods reduced a large book of maps, each of which was an entire skin of parchment, and the whole contained upwards of 46,000 acres, to a pocket volume; and afterward connected all these maps into one map, which was contained in one skin of parchment: therefore, upon the whole I do recommend these methods for reducing maps to be much more accurate than any of the methods commonly used, such as squaring of paper, using a parallelogram, proportionable compasses, or any other method I ever met with, though the figures to be reduced were ever so numerous, irregular, or complicated.

To unite separate maps of lands which join each other into one map of any assigned size.

If there be several large maps contained in a book, each of which suppose to take up a skin of parchment, or a sheet of the largest paper, which maps of lands join each other, and it be required to reduce them to so small a scale that all of them when joined together may be contained in one skin, half a skin, or any assigned sized piece of parchment or paper:—

Having pricked off and copied the several maps on any kind of paper, unite them by cutting with scissors along the edge of one boundary which is adjoining the other, but not cutting by the edge of both, and throw aside the parts cut off: then lay these together on a large table, or on the floor, and where the boundaries agree they will fit in with each other as indentures do; and after this manner they are easily connected: measure then the length and breadth of the entire connected maps, and

the length and breadth of the parchment or paper you are confined to; if the former be three, four, or five times greater (that is, longer and broader) than the latter, reduce each copied map severally to a scale that is three, or four, or five times less, as before; and the same parts of the boundaries you cut by in the large maps, by the same you must also cut in small ones, and unite the small as the large ones were united; cementing them together with white wafer: thus your map will be reduced to the assigned size, which copy over fair on the parchment or paper you were confined to.

But it is not always that a person is confined to a given area of parchment, or paper; in such cases, if there are many large maps to be united into one, reduce each of them severally to a scale of 160 perches to an inch, and unite those by the contiguity or boundaries, as before: or if you have a few, it will be sufficient to reduce them to a scale of 120, &c. But having the maps given, and the scale by which they are laid down, your reason will be sufficient to direct you to know what scale they should be reduced to.

Directions concerning surveys in general.

If you have a large quantity of ground to survey, which consists of many fields or holdings, and that it be required to map and give the respective contents of the same, it is best to make a survey of the whole first, and to be satisfied that it is truly taken, as well as to find its contents; and as you go round the land, to make a note on the side of your field-book at every station where the boundary of any particular field or holding intersects or meets the surround; then proceed from any one of those stations, and in your field-book say, "proceed from such a station," and when you have gone round that field or division, insert the station you close at, and so through the whole: a little practice only can render this sufficiently familiar, and the method of protraction must be evident from the field-notes. When the whole is protracted, and you are satisfied of the closes of the particular divisions, cast up each severally, and if the sum of their contents be equal to the contents of the whole first found, you may safely conclude that all is right.

The protraction being thus finished and cast up, transfer it on clean paper, vellum, or parchment as before; be careful to draw your lines with a fine pen, write on it the names of the circumjacent lands, and set No. 1, 2, 3, 4, &c. in every particular field or division; let every tenant's particular holding be distinguished by a different coloured paint being run finely along the boundaries; let all the roads, rivulets, rivers, bogs, ponds,

houses, castles, churches, beacons, or whatever else may be remarkable on the ground, be distinguished on the map. Write the title of the map in a neat compartment, either drawn or done from a good copper-plate engraving, with the gentleman's arms. Prick off one of your parallels with the map, and on it make a mariner's compass, and draw a flower-de-luce to the north, and this will represent the magnetical north; after which set off the variation, which express in figures, and through the centre of the compass let a true meridian line be drawn of about 3 inches long, by which write True Meridian. Let a scale be drawn, or it is sufficient to express the number of perches to an inch the map was laid down by. Draw a reference table of three or, if occasion be, of four or more columns: in the first insert the number of the field or holding; in the next its name, and by whom occupied; in the third the quantity of acres, roods, and perches it contains; if you have unprofitable land, as bog or mountain, let the quantity be inserted in the fourth column; and if it be required, you may make another column for statute measure, and then the map is completed.

SECTION VII.

THE METHOD OF DIVIDING LAND, OR OF TAKING OFF OR ENCLOSING ANY GIVEN QUANTITY.

EXAMPLE I.

PL. 12. *fig.* 1.

LET *ABCD*, &c. be a map of ground containing 11 acres; it is required to cut off a piece, as *DEFGID*, that shall contain 5 acres.

Join any two opposite stations, as *D* and *G*, with the line *DG* (which you may nearly judge to be the partition line), and find the area of the part *DEFG*, which suppose may want 3R. 20P. of the quantity you would cut off; measure the line *DG*, which suppose to be 70 perches. Divide 3R. 20P., or 140P., by 25, the half of *DG*, and the quotient 4 will be a perpendicular for a triangle whose base is 70 and area 140P. Let *HI* be drawn parallel to *DG*, at the distance of the perpendicular 4, and from *I*, where it cuts the boundary, draw a line to *D*, and that line *DI* will be the division line; or a line from *G* to *H*

will have the same effect; all which must be evident from what has been already said.

But if hills, trees, &c. obstruct the view of the points *D* and *I* from each other, it will be necessary, in order to run a partition line, to know its bearing; and it may be proper on some occasions to have its length; both these may be easily calculated from the common field-notes only, as in the following example, without the trouble of any other measurement on the ground, or any dependence on the map and scale.

EXAMPLE II.

PL. 12. *fig.* 3.

Let *ABCDEFGHIA* be a tract of land to be divided into two equal parts by a right line from the corner *I* to the opposite boundary *CD*; required the bearing and length of the partition line *IN*, by calculation, from the following field-notes, viz.

Field-Notes and Area.		
Bound.	Bearing.	Perch.
AB	N 19° 0′ E	108
BC	S 77 0 E	91
CD	S 27 0 E	115
DE	S 52 0 W	58
EF	S 15 30 E	76
FG	West.	70.9
GH	N 36 0 W	47
HI	North.	64.3
IA	N 62 15 W	59
152A. 1R. 25.9P.		

Operation.

IABCI		Per.	N.	S.	E.	W.	Merid. dist. &c.
IA	N 62¼° W	59	27.5	—	—	52.2	
AB	N 19 E	188	102.1	—	35.2	—	
BC	S 77 E	91	—	20.5	88.7	—	
CI	—— —	—	—	109.1	—	71.7	
Area, 8722.3 perches.			129.6	129.6	123.9	123.9	

152A. 1R. 25.9P. = 24385.9 perches.
Half, to be divided off, = 12192.9 } subtract.
The part *IABCI* = 8722.3 }

Triangle *ICNI* = 3470.6 perches.

ICDI.		Per.	N.	S.	E.	W.	Meridian dist. &c.
IC	N — E	115	109.1	—	71.9	—	
CD	S 27 E	—	—	102.5	52	—	
DI				6.6	—	123.9	
Area 6522.1 per.			109.1	109.1	123.9	123.9	

Then, as { *ICDI* : *CD* : : *ICNI* : *CN* } theo. 18, sec. 1,
{ 6522.1 : 115 : : 3470.6 : 61.19 }
which determines the point *N* in *CD*.

ICNI.		Per.	N.	S.	E.	W.
IC	as before	—	109.1	—	71.7	
CN	S 27° E	61.2	—	54.6	27.8	—
NI			—	—	—	—
	— — —	—	—	54.6	—	99.5

As dif. lat. 54.6
: Radius S. 90°
: : Depart. 99.5
: Tang. Bear. 61° 15′

As S. Bear. 61° 15′
: Depart. 99.5
: : Radius S. 90°
: Distance 113.49

Answer, { *IN* runs N. 61° 15′ E. } 113.5 perches.
{ *NI* runs S. 61 15 W. }

In the part *IABCI*, the difference between the northings and the southings of the three lines *IA*, *AB*, and *BC* (109.1) is the difference of latitude, and that of their eastings and westings (71.7) the departure of the line *CI*, which is placed thereto, so as to balance the columns; see theo. 1, sect 5: hence the contents are obtained, as already taught, without the bearing or length of the line *CI*.

For the triangle *ICDI*, the dif. lat. and dep. of *IC* are taken from the preceding table, which in going from *I* to *C*

will be northing and easting: those of *CD* are found by the bearing and distance, and of *DI* by balancing the columns, as before for *CI*.

The difference of latitude (54.6) and departure (99.5) of the line *NI*, in the third table, are found by balancing those of *IC* and *CN*; and as they are the base and perpendicular of a right-angled triangle, of which the line *NI* is the hypothenuse, and the angle opposite to the departure the bearing, we have the answer by two trigonometrical statings, as above; and thus may any tract be accurately divided, or any proposed quantity readily cut off or enclosed.

Now the student or practitioner may calculate the contents of the part *ABCNIA* (the bearing and distance, or the dif. lat. and dep. of *CN* and of *NI* being known), and if it be found equal to the intended quantity, it proves the truth of the operation.

EXAMPLE III.

PL. 12. *fig.* 3.

It is proposed to cut off 38A. 16¼P. to the south end of this tract, by a line running from *E* due west 40 perches to a well at *O*, and from thence a right line to a point *M* in the boundary *HI*; the place of *M* and the bearing and length of the line *OM* are required, the field-notes being as in example second.

Answer, { *M* from *H*, north 43.23 / *OM*, N. 78° 7′ W. 39.03 } perches.

In this example we find,

The area of *OEFGHO*	=5270.5	
Consequently of *HOMH*	= 826.0	perches.
Dif. lat. of the line *HO*=*HV*	= 35.2	
Departure of ditto =*OV*	= 38.2	

As *HI* happens to be a meridian, the area of *HOMH* divided by half *OV* (19.1) quotes *HM* (42.23), without finding the area of *HOIH*, as we did of *ICDI* in example second, and *HM*—*HV*=*VM*=8.03= dif. lat. of *OM*, which with its dep. *VO*=38.2, gives the bearing and distance as before.

EXAMPLE IV.

PL. 12. *fig.* 4.

A trapezoidal field *ABCD*, bounded as under specified, is to be divided into two equal parts by a right line *EF* parallel to *AB* or *CD*; required *AF* or *BF*.

Bou.	Bearing.	Per.
AB	South.	30
BC	N 80° W	60
CD	N 39½ W	45.5
DA	S 80 E	89.4
13A. 3R. 7P.		

In the triangle *CBG* are given *BC* and all the angles (known by the bearings) to find *BG*, and thence the area by prob. 9, sect. 4. which, + half the area of *ABCD* = area of *EFG*; then, as the area of *CBG* is to that of *EFG*, so is the square of *BG* to the square of *FG*, and *FG*—*BG*=*BF*

Operation at large.

Angle *G* 39° 30′, log. S. Co. Ar.	0.19649	add
Side *BC* 60 per. log.	1.77815	
Angle *C* 40° 30′, sine	9.81254	
Side *BG* 61.26 per.	1.78718	add
Side *BC* 60 per.	1.77815	
Angle *B* 100° 0′, sine	9.99335	
2)3619.8, log.	3.55868	
As *CBG*=1809.9 Co. Ar.	6.74235	add
1103.5=*BCEF*		
Is to *EFG*=2913.4, log.	3.46440	
So is sq. *BG* 61.26, log.	1.78718	
	1.78718	
To sq. *FG* 77.72,	2)3.78111	
Ans. *BF*=16.46 perches	1.89055	

By the application of this method a tract of land may be divided accurately, in any proportion, by a line running in any assigned direction.

Note.—When the practitioner would wish to be very accurate, it will be much better to work by four-pole chains and links than by perches and tenths; one-tenth of a perch square being equal to 6¼ square links.

EXAMPLE V.

The following Field-notes (from A. BURNS*) are of a piece of land, which is proposed, as an example, to be divided into three equal parts by two right lines running from the sixth and seventh stations: and proved by calculating the contents of the middle part.*

St.	Bearing.	4-P. *ch.*
1	N.E. 56¼°	21.60
2	N.E. 26½	13.44
3	S.E. 71½	18.96
4	S.E. 26½	13.44
5	S.W. 71½	18.96
6	S.E. 45	8.47
7	S.E. 63½	13.44
8	N.E. 45	8.47
9	S.E. 26½	13.44
10	S.W. 45	8.47
11	S.W. 63½	13.44
12	N.W. 76	24.73
13	N.W. 36¾	30.00
Area, 167A. 1R. 24P.		

EXAMPLE VI.

PL. 8. *fig.* 5.

The plot *ABCDEFGHA* is proposed to be divided *geometrically*, in the proportion of 2 to 3, by a right line from a given point in any boundary or angle thereof, suppose the point *D*.

Reduce the plot to the triangle *cDe*, as already taught; divide the base *ce* in the point *N*, so that *cN* be to *Ne* in the ratio of 2 to 3, by prob. 14; draw *DN*, and it is done.

EXAMPLE VII.

PL. 12. *fig.* 3.

Example second may likewise be performed geometrically.

Produce *CD* both ways for a base, and reduce the whole to a triangle, making *I* the vertical point; then bisect the base in *N*, and draw *IN*. But,

Notwithstanding this geometrical method is demonstrably true in theory, it is not as safe, on practical occasions requiring accuracy, as the calculation, even when performed with the greatest care; for which reason we will not enlarge on it here.

EXAMPLE VIII.

Suppose 864 acres to be laid out in form of a right-angled parallelogram, of which the sides shall be in proportion as 5 to 3; required their dimensions.

For the greater side, multiply the area by the greater number of the given proportion, and divide by the less, or, for the less side, multiply by the less number, and divide by the greater; the square root of the quotient will be the side required: thus,

864A.=138240P.	1.38240
5	3
3)691200	5)414720
Ans. $\surd$ 230400=480.	$\surd$ 82944=288

EXAMPLE IX.

If it be required to lay out any quantity of ground, suppose 47A. 2R. 16P., in form of a parallelogram, of which the length is to exceed the breadth by a given difference, for instance, 80 perches, then add the square of half this difference to the area, and take the square root of the sum; to which add half the difference for the greater side, and subtract it therefrom for the less: thus,

2)80	47A. 2R. 16P.=7616 perches.
40	1600
40	$\surd$ 9216=96
1600 half diff.;	add and subt. 40

Ans. { the length=136
the breadth=56

Any proposed quantity of ground may be laid out or enclosed in the form

Of a { Square . . . . . by prob. 2d, / Parallelogram, one side given, by prob. 4th, / Triangle of a given base, . by prob. 7th, / Circle . . . . . by prob. 13th, } sec. 4.

It is sometimes most convenient, when land is to be laid out adjacent to a creek, river, or other crooked boundary, to measure offsets to the angles or bending thereof, from a right line or lines taken near such boundary, and to deduct the area of these offsets from the given quantity, and then to lay off the remainder from the right line or lines, in the desired form.

In laying out new lands, attention must be paid to the allowance for roads, as exemplified in prob. 14th.

SECTION VIII.

OF SURVEYING HARBOURS, SHOALS, SANDS, &c.

Pl. 13. *fig.* 1.

There are three methods whereby this may be performed; for the observations may be made either on the water or on the land. Those made on the water are of two kinds; one by the log-line and compass (as in plane sailing measuring) the course and distance round the sand; and then to be plotted as a large wood, or any enclosure taken by the circumferentor.

This method I omit, for two reasons: first, because it is to be deduced from the writers of navigation; and, secondly, because the distances thus measured are liable to the errors of currents, which generally attend shoals or sands near the shore.

The second method, when there are no distances to be measured on the water, though still there is one inconvenience, common also to the former, because the bearings or observations are to be taken on that unstable element (an error scarce mentioned by practical artists), I shall briefly hint at; and so rather choose a third, which is liable to neither of these imperfections.

Let a boat be manned out with a signal flag, a log and line, lead and line, and, to observe the bearings of any landmark, a compass with sights.

Take two or more objects or places, as *A*, *B*, *C*, on the

shore, from whence the boat may be seen on the several parts of this shoal, and determine their relative position by bearing and distances either before or after the other necessary observations are made.

One of the boat's crew is to sound till he finds himself on the edge of the sand, by the depth of water, and then to come to an anchor; which he is to signify to two persons on the shore at *B* and *C*, by his signal. And then from those known landmarks *B* and *C* the observers are to take the bearings of the boat, and to register their observations; which, when done, they are to signify to the crew by waving a flag, or by some other signal.

And in the mean time, to prevent mistakes, let the crew take the bearings of each of these landmarks: then weigh anchor, which suppose at *D*.

Then, by sounding, proceed to *E*, and make like observations. And so at *E*, *F*, *G*, &c., till you have surrounded your sand.

And if in the process you are about to lose the sight of one of your landmarks, suppose *C*, let your assistant at *C*, or *B*, who at that time will also be about to lose the sight of the boat, by signals (before agreed on) remove to some other object beforehand agreed on, suppose to *H* or *K*, and then to proceed as before.

Lastly, if the sand runs so far out at sea that the object cannot be seen by the boat, nor the boat by the observer on shore, there may be rockets fired by the boat's crew, and also by the observers on the shore in the night, whereby those bearings may be taken almost at as great a distance as the light can be seen. For supposing they rise but a quarter of a mile above the apparent horizon, its stay will be about 9 seconds, and its distance for this quarter of a mile will be visible about 44 miles.

But rockets rise much higher, and then the distances are much greater whereby they are visible.

Or two boats may lay at anchor, instead of the landmarks, and then you may work as before.

Now since the landmarks *B* and *C* are fixed, their position may be laid down in the draught, as in common surveying, by plotting the distance between *B* and *C*. And then by plotting the line *BD*, and the line *DC*, according to their position, their common intersection will give the point *D*. And in like manner *E*, *F*, *G*, &c. may be plotted; and so the shoals completed. And this from the bearings taken at *B* and *C*.

If this be a standing lake, environed by bogs, or other impediments, the observations at *D*, *E*, *F*, &c., by taking their

opposites, may suffice to plot the same from the landmarks, *A*, *B*, *C*, &c. as well as those taken on the land: or, indeed, by the course and distance, as in navigation, if the water be smooth and without a current.

In sea shoals, it is convenient to note at each observation the depth of the water found by the lead, and the drift and setting of the current by the log and compass, while the boat is at anchor, which may be done with ease and expedition enough. For while the boat rides at an anchor, her stern points out the setting of the current, and the log and glass will measure its drift.

And these ought to be noted on the draught, which may be thus:

The currents may be shown, by drawing a dart pointing out its setting, and its drift by the Roman capital letters, the depth of the water by the small figures, and rocks by little crosses, &c.

SECTION IX.

OF LEVELLING.

Pl. 13. *fig.* 2.

Levelling is the art of ascertaining the perpendicular ascent or descent of one place (or more) above or below the horizontal level of another, for various intentions, and of marking out courses for conveyance of water, &c.

The *true level* is a curve conforming to the surface of the earth; as *ABG*.

The *apparent level* is a tangent to that curve; as *ADE*.

The *correction* or allowance for the earth's curvature is the difference between the apparent level and the true; as *BD*. The quantity of this correction may be known by having in the right-angled triangle *CAD* the two legs *AC* = the semidiameter of the earth (=1267500 perches), and *AD* = the distance of the object, to find the hypothenuse *CD*, from which taking *CB* (=*CA*), the remainder will be the correction *BD*; but it may be obtained more practically thus:

Square the distance in { four-pole chains, and divide by 800, / or in perches, and divide by 12800 / or in miles, and multiply by 8 } for the correction in inches.

EXAMPLE.

Required the correction for 20 four-pole chains = 80 perches = $\frac{1}{4}$ mile.

$$800)20\times20=400(.5$$
$$12800)80\times80=6400(.5$$

$\frac{1}{4}=.25$, and $.25\times.25\times8=.5$.

That is, .5, or $\frac{1}{2}$ inch, the correction required.

But, to save the trouble of calculation, we insert the following table of corrections.

A Table of Corrections.

The distances in four-pole chains.

Distan.	Correc.	Distan.	Correc.
Chains.	Inches.	Chains.	Inches.
1	0.00125	27	0.91
2	0.005	28	0.98
3	0.01125	29	1.05
4	0.02	30	1.12
5	0.03	31	1.19
6	0.04	32	1.27
7	0.06	33	1.35
8	0.08	34	1.44
9	0.10	35	1.53
10	0.12	36	1.62
11	0.15	37	1.71
12	0.18	38	1.80
13	0.21	39	1.91
14	0.24	40	2.00
15	0.28	45	2.28
16	0.32	50	3.12
17	0.36	55	3.78
18	0.40	60	4.50
19	0.45	65	5.31
20	0.50	70	6.12
21	0.55	75	7.30
22	0.60	80	8.00
23	0.67	85	9.03
24	0.72	90	10.12
25	0.78	95	11.28
26	0.84	100	12.50

The first thing necessary in levelling is the adjusting of the level, which may be performed several ways. The following is very easy and practical.

Choose some ground which is not above 4 or 5 feet out of the level, for the distance of 8 or 10 chains in length, and suppose it be AB (fig. 3), and find the middle between A and B, which suppose to be C; plant the intstrument at C, direct the tube to a station-staff held up at A, and elevate or depress the tube till the bubble is exactly in the middle of the divisions; then by signals direct your assistant at A to raise or depress the vane sliding on the station-staff till the horizontal hair in the glass cuts the middle of that vane; then see how many feet, inches, and parts are cut by the upper part of the vane, which suppose to be 3 feet 4 inches and 6 tenths.

In like manner direct to the other staff at B, and suppose the upper edge of that vane to be cut at the height of 6 feet 5 inches and 2 tenths, then will these two vanes be on a level.

From 6 feet 5.2 inches subtract 3 feet 4.6 inches, and reserve the remainder 3 feet 0.6 inches.

Now remove the instrument as close to the higher station-staff as you can; so that the middle of the telescope may almost touch it. Then bring the telescope as near to a level as the judgment of the eye will direct.

Measure from the ground the height of the top of the telescope; and also of the bottom in feet, inches, and parts; suppose them to be 4 feet 10.5 inches, and 5 feet 0.3 inches; then half the sum of the heights 4 feet 11.4 inches is the height of the centre of the glass; and to this add half the breadth of the vane, which suppose to be 1 inch and 5 tenths, and to the sum 5 feet 0.9 inches add the preceding remainder 3 feet 0.6 inches; then let the person at B move his vane till the upper edge cut 8 feet 1.5 inches, the sum of the preceding numbers.

Now so elevate or depress the hair of the bubble till the hair cut the middle of the vane at B, and at the same time the bubble stands at the middle of the divisions; and then will the instrument be duly adjusted.

If you have a mind to be more accurate, repeat the operation; but when you place the instrument at C, turn the tube at right angles to the line AB, and there set it level; then proceed with a repetition of the work. Only observe to cross-level it in this adjustment, and in all future uses whatsoever.

Or the level may be adjusted thus: As before, first plant the instrument in the middle between A and B (fig. 4), and observe the heights on the station-staves, which suppose to be as above; and consequently their difference, as before, is 3 feet 0.6 inches. Now measure from C towards the highest ground

A, some distance that comes almost to *A*, suppose 4 chains to *D*; and *DB* will be 9 chains, and *DA* one chain; then plant the instrument at *D*, direct the telescope to *A*, and setting the bubble to the middle of the division direct your assistant to move the vane till the hair cuts the middle of it, and note down the feet, inches, and parts cut by the upper edge of the vane, which suppose to be 3 feet 8.4 inches: to this add the difference 3 feet 0.6 inches, and the sum 6 feet 9 inches reserve.

Now direct the telescope to the staff at *B*, level it, and direct your assistant to move the vane till the hair cuts the middle thereof; and then if the upper edge of the vane cuts the foregoing sum 6 feet 9 inches, the hair and bubble are truly adjusted. But if not, say, as *BD* less *AD* is to the difference between the numbers cut by the upper edge of the vane and the number 6 feet 9 inches, so is the distance *AD* to a number which, added to that cut by the vane when less than 6 feet 9, and subtracted from the number cut by the vane when it is greater than 6 feet 9, will give a number, to which let the assistant fix the vane; then so elevate or depress the hair or the bubble till the hair cuts the middle of the vane at *B*, and the bubble stands in the middle of the divisions; for then the level will be adjusted. The operation may be again repeated, and at every station cross-levelled, which will confirm the former adjustment.

Or it will be still better to set the station-staves equally distant from the instrument (suppose about 16 or 20 perches each) at an angle of about 60°, or so as to form nearly an equilateral triangle therewith, and level the two vanes (*A* and *B*, fig. 5), as before, which will be then both in the same horizontal level, whether the instrument be rightly adjusted or not, because one will be as much above or below the true level of the instrument as the other, being in the same distance from it; then remove the instrument as near as may be to one of them, suppose *A*, and raise or lower the vane *A* to the exact level of the visual ray in the instrument, noting precisely how much it is moved, and have the other vane *B* moved just as much, in order to bring them again to a level, allowing for the correction of the apparent level if it be a sensible quantity; then adjust the instrument to the level of the vane at *B*.

To adjust the rafter-level (plate 13, fig. 6), which may be 10, 12, or 14 feet in the span *AB*; set it on a plank or hard ground nearly level, and mark where the plumb line cuts the beam *mn*, suppose at *c*; then invert the position by setting the foot *A* in the place of *B*, and *B* in that of *A*, marking where the line now cuts, as at *e*; the middle point between *c* and *e* will be the true levelling mark.

To continue a level course with this instrument, set the foot *A* to the starting place, and move *B* upward or downward towards *D* or *E*, till the point *B* be determined and marked for a level with *A*; then carry the instrument forward in the direction of *C*, till the foot *A* rests at *B*, whence the point *C* is levelled as before, &c. Sights may be placed at *r* and *s*, and the instrument adjusted to them, as before, by reversing them in the direction of some distant object.

After the instrument is duly adjusted, you may proceed to use it. Let the example be this annexed (fig. 7), where *A* everywhere represents the level, and *B* the station-staves; and suppose the route be made from *a* to *e*: first plant the instrument between the staves *a* and *b*; at *A* direct the level to *aB*, bring the bubble to the middle of the divisions, and instruct your assistant so to place the vane that the hair in the telescope cuts the middle of the vane; then in a book divided into two columns, the one entitled *Back-sights*, the other *Fore-sights*, enter the feet, inches, and parts cut by the upper edge of the vane at *aB* in the column entitled *Back-sights*.

Then look towards the other staff *bB*, bring the bubble to the middle of the divisions, and direct your assistant to place the vane so that the hair cuts the middle of the vane; then enter the feet, inches, and parts cut by the upper edge of the vane in the column of *Fore-sights*.

Now plant the instrument at A^2, still keeping the staff *Bb* exactly in the same place, and carry the staff *aB* forward to the place *cB*; now look back to the staff *bB*, and enter the numbers cut by the vane there under the title *Back-sights*; then look forwards to *cB*, and enter the observation under the title *Fore-sights*. Do the like when the instrument is planted at A^3, A^4, &c., always taking care to keep the staff in the same place when you looked at it for a *fore-sight*, till you have also taken with it a *back-sight*.

Having finished your level, add up the column of *back-sights* into one sum, and the column of *fore-sights* also into one sum; and the difference between these sums is the ascent or descent required. And if the sum of the *fore-sights* be greater than the sum of the *back-sights*, then *e* is lower than *a*; but if the sum of the *fore-sights* be less than the sum of the *back-sights*, *e* is higher than *a*. For example, let the numbers be as in the following table.

Back-sights.			*Fore-sights.*		
Feet.	Inches.	Tenths.	Feet.	Inches.	Tenths.
3	7	5	6	4	5
4	6	8	8	3	2
6	0	2	5	4	7
9	5	0	8	7	8
1	0	7	9	4	8
24	8	2	38	1	0
			24	8	2
Hence the descent is			13	4	8
			13	4	8

Observations.

1. And if the distances thus taken are short, the curvature of the earth may be rejected. For, if the distance from the instrument be everywhere about 100 yards, all the curvatures in a mile's work will be less than half an inch.

2. If the distance from the instrument to the hindermost staff be everywhere equal to the distance from the instrument to the corresponding staff, the curvature of the earth and the minute errors of the instrument will both be destroyed. Hence it will be much better to set the instrument as equally distant from both staves as may be.

3. If the distances of the instrument from the staves be very unequal and very long, the curvatures must be accounted for, and the distances, in order thereto, must be measured.

4. Therefore it appears, that the best method to take a level is, to measure the several distances from the instrument to the back and forward station-staves; and enter them in the field-book, according to the titles of their several columns, as in the following example; and correct the heights from the table of allowances, which may be done at home when you are about to sum up the heights.

Backwards.			Forwards.		
Distan.	Height.	Corrected.	Distan.	Height.	Corrected.
Links.	Inches.	Inches.	Links.	Inches.	Inches.
370	3.25	3.24	418	4.36	4.34
420	6.10	6.08	328	7.18	7.17
760	5.38	5.31	289	6.75	6.67
584	7.25	7.21	530	9.53	9.50
326	8.15	8.14	485	11.25	11.22
658	10.25	10.20	376	8.65	8.63
530	6.32	6.29	720	10.34	10.28
36.58		46.47	31.46		57.81
31.46					46.47
68.04					11.34

So that the fall in 68 chains is about 11 inches and $\frac{1}{3}$ of an inch.

Lastly, though hitherto we have considered the level with one telescope only, the same observations may be applied to a level with a double telescope; and I would advise those who use the double telescope, at every station to turn that end of the telescope forward which before was the contrary way.

A more general method of levelling, adapted to the surveying of roads and hilly ground, is exhibited in the following example, in which the measures are given in links.

EXAMPLE.

Pl. 13. *fig.* 8.

Required the bearing and distance of the place *B* from *A*, and its perpendicular ascent or descent above or below the horizontal level of *A*.

St.	Course or Bearing.	Elevation or Depression.	Obl. Dist.	Hor. Dist.	Perpen. Ascent or Des.	Dif. Lat.	Depart.
1	N.E. 79° 15′	D. 17° 15′	738	705	218.9	131	692
2	N.E. 75 00	D. 21 45	684	635	253.4	164	613
3	N.E. 50 30	E. 14 00	976	947	236.1	602	730
4	S.E. 85 15	D. 11 30	930	911	185.4	75	908
5	S.E. 70 00	E. 19 15	620	585	204.0	200	549
			3948	3783	217.6 Desc.	622 N.	3492 E.

As dif. lat. 622
Is to radius S. 20°,
So is dep. 3492
To T. bear. 79° 54′.

As S. bear. 79° 45′
Is to dep. 3492,
So is radius S. 90°
To dist. 3547.

As 100 links : 66 feet : : 217.6 links : 143.6 feet, the descent *B* below the level of *A*.

Hence, *B* bears N. 79° 54′ E. from *A*.
Nearest horiz. dist. 3547 links.
Sum of obl. dist. 3948 links.
Sum of horiz. dist. 3783 links.
Perp. desc. 217.6 links = 143.6ft.
} Answer.

With the angular elevation or depression in the third column, and the oblique distance in the fourth (as course and distance) are found the horizontal distance in the fifth, and the perpendicular ascent or descent on the sixth, for each station (as difference of latitude and departure): then, with the bearing and horizontal distance, we get the difference of latitude and departure in the last two columns.

The ascents and descents in the sixth column are distinguished by the letters E and D in the third, signifying elevation or depression; and being added separately, the difference of their sums is set at the bottom of the column with the name of the greater, and shows the perpendicular descent of *B* below the horizontal level of *A*.

In like manner the northings and southings in the seventh column are distinguished by the letters N and S in the second, &c.

PROMISCUOUS QUESTIONS.

1. The perambulator, or surveying wheel, is so contrived as

to turn just twice in the length of a pole, or 16½ feet; what then is the diameter? Ans. 2.626 feet.

2. Two sides of a triangle are respectively 20 and 40 perches; required the third, so that the contents may be just an acre. Ans. either 23.099 or 58.876 perches.

3. I want the length of a line by which my gardener may strike out a round orangery that shall contain just half an acre of ground. Ans. 27¾ yards.

4. What proportion does the arpent of France, which contains 100 square poles of 18 feet each, bear to the American acre, containing 160 square poles of 16.5 feet each, considering that the length of the French foot is to the American as 16 to 15? Ans. as 512 to 605.

5. The ellipse in Grosvenor Square measures 840 links the longest way, and 612 the shortest, within the rails: now the wall being 14 inches thick, it is required to find what quantity of ground it encloses, and how much it stands upon.

Ans. It encloses 4A. 6P., and stands on 1760¼ square feet.

6. Required the dimensions of an elliptical acre with the greatest and least diameters in the proportion of 3 to 2.

Ans. 17.479 by 11. 653 perches.

7. The paving of a triangular court at 18*d.* per foot, came to 100*l.* The longest of the three sides was 88 feet: what then was the sum of the other two equal sides?

Ans. 106.85 feet.

8. In 110 acres of statute measure, in which the pole is 16½ feet, how many Cheshire acres, where the customary pole is 6 yards, and how many of Ireland, where the pole in use is 7 yards?

Ans. 92A. 1R. 28P. Cheshire; 67A. 3R. 25 P. Irish.

9. The three sides of a triangle containing 6A. 1R. 12P. are in the ratio of the three numbers 9, 8, 6, respectively; required the sides. Ans. 59.029, 52.47, and 39.353.

10. In a pentangular field, beginning with the south side, and measuring round towards the east, the first or south side is 2735 links, the second 3115, the third 2370, the fourth 2925, and the fifth 2220; also the diagonal from the first angle to the third is 3800 links, and that from the third to the fifth 4010; required the area of the field. Ans. 117A. 2R. 28P.

11. Required the dimensions of an oblong garden containing three acres, and bounded by 104 perches of pale fence.

Ans. 40 perches by 12.

12. How many acres are contained in a square meadow, the diagonal of which is 20 perches more than either of its sides?

Ans. 14A. 2R. 11P.

13. If a man six feet high travel round the earth, how much greater will be the circumference described by the top of his head than by his feet? Ans. 37.69 feet.

N. B.—The required difference is equal to the circumference of a circle 6 feet radius, let the magnitude of the earth be what it may.

14. Required the dimensions of a parallelogram containing 200 acres, which is 40 perches longer than wide.

Ans. 200 perches by 160.

15. What difference is there between a lot 28 perches long by 20 broad, and two others, each of half the dimensions?

Ans. 1A. 3R.

PART III.

Containing the astronomical methods of finding the latitude, variation of the compass, &c., with a description of the instruments used in these operations.

SECTION I.

INTRODUCTORY PRINCIPLES.

Day and night arise from the circumrotation of the earth. That imaginary line about which the rotation is performed is called the axis, and its extremities are called poles. That towards the most remote parts of Europe is called the *north pole*, and its opposite the *south pole*. The earth's axis being produced will point out the *celestial poles*.

The equator is a great circle on the earth, every point of which is equally distant from the poles; it divides the earth into two equal parts, called hemispheres: that having the north pole in its centre is called the *northern hemisphere*, and the other the *southern hemisphere*. The plane of this circle being produced to the fixed stars will point out the celestial equator, or equinoctial. The equator, as well as all other great circles of the sphere, is divided into 360 equal parts, called *degrees*; each degree is divided into 60 equal parts, called *minutes*; and the sexagesimal division is continued.

Note.—The ancients, having no instruments by which they could make observations with any tolerable degree of accuracy, supposed the length of the year, or annual motion of the earth, to be completed in 360 days: and hence arose the division of

the circumference of a circle into the same number of equal parts, which they called *degrees*.

The meridian of any place is a semicircle passing through that place, and terminating at the poles of the equator. The other half of this circle is called the *opposite meridian*.

The latitude of any place is that portion of the meridian of that place which is contained between the equator and the given place; and is either *south* or *north*, according as the given place is in the northern or southern hemisphere, and therefore cannot exceed 90°.

The parallel of latitude of any place is a circle passing through that place parallel to the equator.

The difference of latitude between any two places is an arch of a meridian intercepted between the corresponding parallels of latitude of those places. Hence, if the places lie between the equator and the same pole, their difference of latitude is found by subtracting the less latitude from the greater; but if they are on opposite sides of the equator, the difference of latitude is equal to the sum of the latitudes of both places.

The first meridian is an imaginary semicircle, passing through any remarkable place, and is therefore arbitrary. Thus, the British esteem that to be the first meridian which passes through the royal observatory at Greenwich; and the French reckon for their first meridian that which passes through the royal observatory at Paris.—Formerly many French geographers reckoned the meridian of the island of Ferro to be their first meridian; and others, that which was exactly 20 degrees to the west of the Paris observatory. The Germans, again, considered the meridian of the Peak of Teneriffe to be the first meridian. By this mode of reckoning, Europe, Asia, and Africa are in east longitude, and North and South America in west longitude. At present the first meridian of any country is generally esteemed to be that which passes through the principal observatory, or chief city, of that country.

The longitude of any place is that portion of the equator which is contained between the first meridian and the meridian of that place; and is usually reckoned either *east* or *west*, according as the given place is on the east or west side of the first meridian; and, therefore, cannot exceed 180°.

The difference of longitude between any two places is the intercepted arch of the equator between the meridians of those places, and cannot exceed 180°.

There are three different horizons, the apparent, the sensible, and the true. The apparent or visible horizon is the utmost apparent view of the sea or land; the sensible is a plane

passing through the eye of an observer, perpendicular to a plumb-line hanging freely; and the true or rational horizon is a plane passing through the centre of the earth, parallel to the sensible horizon.

Altitudes observed at sea are measured from the visible horizon. At land, when an astronomical quadrant is used, or when observations are taken with a Hadley's quadrant by the method of reflection, the altitude is measured from the sensible horizon; and in either case the altitude must be reduced to the true horizon.

The zenith of any given place is the point immediately above that place, and is, therefore, the elevated pole of the horizon. The nadir is the other pole, or point diametrically opposite.

A vertical is a great circle passing through the zenith and nadir; and therefore intersecting the horizon at right angles.

The altitude of any celestial body is that portion of a vertical which is contained between its centre and the true horizon. The meridian altitude is the distance of the object from the true horizon, when on the meridian of the place of observation. When the observed altitude is corrected for the depression of the horizon and the errors arising from the instrument, it is called the *apparent altitude;* and when reduced to the true horizon, by applying the parallax in altitude, it is called the *true altitude.* Altitudes are expressed in degrees and parts of a degree.

The zenith distance of any object is its distance from the zenith, or the complement of its altitude.

The declination of any object is that portion of its meridian which is contained between the equinoctial and the centre of the object; and is either north or south according as the star is between the equinoctial and the north or south pole.

The ecliptic is that great circle in which the annual revolution of the earth round the sun is performed. It is so named because eclipses cannot happen but when the moon is in or near that circle. The inclination of the ecliptic and equinoctial is at present about 23° 28′; and by comparing ancient with modern observations, the obliquity of the ecliptic is found to be diminishing—which diminution, in the present century, is about half a second yearly.

The ecliptic, like all other great circles of the sphere, is divided into 360°; and is further divided into twelve equal parts, called signs: each sign, therefore, contains 30°. The names and characters of these signs are as follows:

Aries, ♈	Cancer, ♋	Libra, ♎	Capricornus, ♑
Taurus, ♉	Leo, ♌	Scorpio, ♏	Aquarius, ♒
Gemini, ♊	Virgo, ♍	Sagittarius, ♐	Pisces, ♓

Since the ecliptic and equinoctial are great circles, they therefore bisect each other in two points, which are called the *equinoctial points.* The sun is in one of these points in March, and in the other in September; hence, the first is called the *vernal*, and the other the *autumnal* equinox—and that sign which begins at the vernal equinox is called *Aries.* Those points of the ecliptic which are equidistant from the equinoctial points are called the *solstitial points;* the first the *summer*, and the second the *winter solstice.* That great circle which passes through the equinoctial points and the poles of the earth is called the *equinoctial colure;* and the great circle which passes through the solstitial points and the poles of the earth is called the *solstitial colure.*

When the sun enters Aries it is in the equinoctial, and therefore has no declination. From thence it moves forward in the ecliptic, according to the order of the signs, and advances towards the north pole, by a kind of retarded motion, till it enters Cancer, and is then most distant from the equinoctial; and moving forward in the ecliptic, the sun apparently recedes from the north pole with an accelerated motion till it enters Libra, and, being again in the equinoctial, has no declination; the sun, moving through the signs Libra, Scorpio, and Sagittarius, enters Capricorn; and then its south declination is greatest, and is therefore most distant from the north pole; and moving forward through the signs Capricorn, Aquarius, and Pisces, again enters Aries: hence a period of the seasons is completed, and this period is called a solar year.

The signs Aries, Taurus, Gemini, Cancer, Leo, and Virgo are called *northern signs*, because they are contained in that part of the ecliptic which is between the equinoctial and north pole; and, therefore, while the sun is in these signs, its declination is *north:* the other six signs are called *southern signs.* The signs in the first and fourth quarters of the ecliptic are called *ascending signs*, because while the sun is in these signs it approaches the north pole; and, therefore, in the northern, temperate, and frigid zones, the sun's meridian altitude daily increases; or, which is the same, the sun ascends to a greater height above the horizon every day. The signs in the second and third quarters of the ecliptic are called *descending signs.*

The tropics are circles parallel to the equinoctial, whose distance therefrom is equal to the obliquity of the ecliptic.

The northern tropic touches the ecliptic at the beginning of Cancer, and is therefore called the *tropic of Cancer;* and the southern tropic touches the ecliptic at the beginning of Capricorn, and is hence called the *tropic of Capricorn.*

Circles about the poles of the equinoctial, and passing through the poles of the ecliptic, are called polar circles; the distance, therefore, of each polar circle from its respective pole is equal to the inclination of the ecliptic and equinoctial. That circle which circumscribes the north pole is called the *arctic* or *north polar circle;* and that towards the south pole, the *antarctic* or *south polar circle.*

That semicircle which passes through a star, or any given point of the heavens, and the poles of the ecliptic, is called a circle of latitude.

The reduced place of a star is that point of the ecliptic which is intersected by the circle of latitude passing through that star.

The latitude of a star is that portion of the circle of latitude contained between the star and its reduced place; and is either *north* or *south,* according as the star is between the ecliptic and the north or south pole thereof.

The longitude of a star is that portion of the ecliptic contained between the vernal equinox and the reduced place of the star.

SECTION II.

Description of the instruments requisite in astronomical observations.

THE QUADRANT.

It is generally allowed that we are indebted to John Hadley, Esq. for the invention, or at least for the first public account, of that admirable instrument commonly called Hadley's quadrant, who in the year 1731 first communicated its principles to the Royal Society, which were by them published soon after in their Philosophical Transactions; before this period the cross-staff and Davis's quadrant were the only instruments used for measuring altitudes at sea, both very imperfect, and liable to considerable error in rough weather; the superior excellence, however, of Hadley's quadrant soon obtained its

general use among seamen, and the many improvements this instrument has received from ingenious men at various times have rendered it so correct, that it is now applied, with the greatest success, to the important purposes of ascertaining both the latitude and longitude at sea or land.

Figure 2, Frontispiece, represents a quadrant of reflection, the principal parts of which are, the octant or frame *ABC* (which is generally made of ebony, or other hard wood, and consists of an arch firmly attached to two radii or bars, which are strengthened and bound by the two braces in order to prevent it from warping), the graduated arch *BC*, the index *D*, the nonius or vernier scale *E*, the index glass *F*, the horizon glasses *G* and *H*, the dark glasses or screens *I*, and the sight vanes *K* and *L*.

The arch, or limb *BC*, although only the eighth part of a circle, is, on account of the double reflection, divided into 90 degrees, numbered 0, 10, 20, 30, &c., from the right towards the left: these are subdivided into three parts, containing each 20 minutes, which are again subdivided into single minutes, by means of a scale at the end of the index. The arch extending from 0 towards the right-hand is called the *arch of excess*.

The index *D* is a flat brass bar, that turns on the centre of the instrument; at the lower end of the index there is an oblong opening; to one side of this opening a nonius scale is fixed, to subdivide the divisions of the arch; at the bottom or end of the index there is a piece of brass which bends under the arch, carrying a spring to make the nonius scale lie close to the divisions; it is also furnished with a screw to fix the index in any desired position.

Some instruments have an *adjusting* or *tangent-screw*, fitted to the index, that it may be moved more slowly, and with greater regularity and accuracy than by the hand; it is proper, however, to observe, that the index must be previously fixed near its right position by the above-mentioned screw, before the adjusting screw is put in motion.

The nonius is a scale fixed to the end of the index, for the purpose, as before observed, of dividing the subdivisions on the arch into minutes; it sometimes contains a space of 7 degrees, or 21 subdivisions of the limb, and is divided into 20 equal parts; hence each division on the nonius will be one-twentieth part greater, that is, one minute longer, than the divisions on the arch; consequently, if the first division of the nonius, marked 0, be set precisely opposite to any degree, the relative position of the nonius and the arch must be altered one minute, before the next division on the nonius will coincide

with the next division on the arch, the second division will require a change of two minutes, the third of three minutes, and so on, till the 20th stroke on the nonius arrives at the next 20 minutes on the arch; the 0 on the nonius will then have moved exactly 20 minutes from the division whence it set out, and the intermediate divisions of each minute have been regularly pointed out by the divisions of the nonius.

The divisions of the nonius scale are in the above case reckoned from the middle towards the right, and from the left towards the middle; therefore the first 10 minutes are contained on the right of the 0, and the other 10 on the left. But this method of reckoning the divisions being found inconvenient, they are more generally counted beginning from the right-hand towards the left; and then 20 divisions on the nonius are equal to 19 on the limb, consequently one division on the arch will exceed one on the nonius by one-twentieth part, that is, one minute.

The 0 on the nonius points out the entire degrees and odd twenty minutes subtended by the objects observed; and if it coincides with a division on the arch, points out the required angle: thus, suppose the 0 on the nonius stands at 25 degrees, then 25 degrees will be the measure of the angles observed; if it coincides with the next division on the left-hand, 25 degrees 20 minutes is the angle; if with the second division beyond 25 degrees, then the angle will be 25 degrees 40 minutes; and so on in every instance where the 0 on the nonius coincides with a division on the arch; but if it does not coincide, then look for a division on the nonius that stands directly opposite to one on the arch, and that division on the nonius gives the odd minutes to be added to that on the arch nearest the right-hand of the 0 on the nonius; for example, suppose the index division does not coincide with 25 degrees, but that the next division to it on the nonius is the first coincident division, then is the required angle 25 degrees 1 minute; if it had been the second division the angle would have been 25 degrees 2 minutes, and so on to 20 minutes, when the 0 on the nonius would coincide with the first 20 minutes on the arch from 25 degrees. Again, let us suppose the 0 on the nonius to stand between 50 degrees and 50 degrees 20 minutes, and that the 15th division on the nonius coincides with a division on the arch, then is the angle 50 degrees 15 minutes. Further, let the 0 on the nonius stand between 45 degrees 20 minutes and 45 degrees 40 minutes, and at the same time the 14th division on the nonius stands directly opposite to a

division on the arch, then will the angle be 45 degrees 34 minutes.

The index glass *F* is a plane speculum, or mirror of glass quicksilvered, set in a brass frame, and so placed that the face of it is perpendicular to the plane of the instrument, and immediately over the centre of motion of the index. This mirror being fixed to the index moves along with it, and has its direction changed by the motion thereof.

This glass is designed to reflect the image of the sun, or any other object, upon either of the two horizon glasses, from whence it is reflected to the eye of the observer. The brass frame, with the glass, is fixed to the index by the screw *M*; the other screw *N* serves to place it in a perpendicular position, if by any accident it has been put out of order.

The horizon glasses *G* and *H* are two small speculums on the radius of the octant; the surface of the upper one is parallel to the index glass when the 0 on the nonius is at 0 on the arch; these mirrors receive the rays of the object reflected from the index glass, and transmit them to the observer. The fore horizon glass *G* is only silvered on its lower half, the upper half being transparent, in order that the direct object may be seen through it. The back horizon glass *H* is silvered at both ends; in the middle there is a transparent slit, through which the horizon may be seen. Each of these glasses is set in a brass frame, to which there is an axis; this axis passes through the wood-work, and is fitted to a lever on the under side of the quadrant, by which the glass may be turned a few degrees on its axis, in order to set it parallel to the index glass.

To set the glasses perpendicular to the plane of the quadrant there are two sunk screws, one before and one behind each glass: these screws pass through the plate on which the frame is fixed into another plate, so that by loosening one and tightening the other of these screws, the direction of the frame, with its mirror, may be altered, and thus be set perpendicular to the plane of the instrument.

The dark glasses, or shades, *I*, are used to prevent the bright rays of the sun, or the glare of the moon, from hurting the eye at the time of observation; there are generally three of them, two red, and one green. They are each set in a brass frame which turns on a centre, so that they may be used separately or together, as the brightness of the object may require. The green glass may be used also alone, if the sun be very faint; it is likewise used in taking observations of the moon; when these glasses are used for the fore observation, they are set

immediately before the fore horizon glass, as in fig. 1, but in front of the other horizon glass at *O* when a back observation is made.

The sight vanes *K* and *L* are pieces of brass, standing perpendicular to the plane of the instrument: the vane *K* is called the *fore sight vane*, and *L the back sight vane*. There are two holes in the fore sight vane, the lower of which and the upper edge of the silvered part of the fore horizon glass are equidistant from the plane of the instrument, and the other is opposite to the middle of the transparent part of that glass; the back sight vane has only one hole, which is exactly opposite to the middle of the transparent slit in the horizon glass to which it belongs: but as the back observations are liable to many inconveniences and errors, we shall not give any directions for their practice.

The adjusting lever (fig. 3), which is fixed on the back of the quadrant, serves to adjust the horizon glass, by placing it parallel to the index glass; when this lever is to be used, the screw *B* must be first loosened, and when by the adjuster *A*, the horizon glass is sufficiently moved, the screw *B* must be fastened again, by which means the horizon glass will be kept from changing its position.

ADJUSTMENTS.

The several parts of the quadrant being liable to be out of order from a variety of accidental circumstances, it is necessary to examine and adjust them, so that the instrument may be put into a proper state previous to taking observations.

An instrument properly adjusted must have the index glass and horizon glasses perpendicular to the plane of the quadrant; the plane of the fore horizon glass parallel, and that of the back horizon glass perpendicular, to the plane of the index glass, when the 0 on the nonius is at 0 on the arch; hence, the quadrant requires five adjustments, the first three of which, being once made, are not so liable as the last two to be out of order; however, they should all be occasionally examined, in case of an accident.

I. *To set the plane of the index glass perpendicular to that of the instrument.*

Place the index near to the middle of the arch, and holding the quadrant in a horizontal position, with the index glass close to the eye, look obliquely down the glass, in such a manner that you may see the arch of the quadrant by direct view and by reflection at the same time; if they join in one direct line,

and the arch seen by reflection forms an exact plane, or straight line, with the arch seen by direct view, or if the image of any point of the arch near *B* appear of the same height as the corresponding part of the arch near *C*, seen direct, the glass is perpendicular to the plane of the quadrant; if not, it must be restored to its right position by loosening the screw *M*, and tightening the screw *N*, or *vice versa*, by a contrary operation.

II. *To set the fore horizon glass parallel to the index glass, the index being at* 0.

Set the 0 on the nonius exactly against 0 on the arch, and fix it there by the screw at the under side. Then holding the quadrant vertically, with the arch lowermost, look through the sight vane, at the edge of the sea, or any other well-defined and distant object. Now, if the horizon in the silvered part exactly meets, and forms one continued line with that seen through the unsilvered part, the horizon glass is parallel to the index glass. But if the horizons do not coincide, then loosen the button-screw in the middle of the lever, on the under side of the quadrant, and move the horizon glass on its axis, by turning the nut at the end of the adjusting lever, till you have made them perfectly coincide; then fix the lever firmly in this situation by tightening the button-screw. This adjustment ought to be repeated before and after every observation. Some observers adopt the following method, which is called finding the *index error*. Let the horizon glass remain fixed, and move the index till the image and object coincide; then observe whether 0 on the nonius agrees with 0 on the arch, if it does not, the number of minutes by which they differ is to be added to the observed altitude or angle, if the 0 on the nonius be to the right of the 0 on the arch, but if to the left of the 0 on the limb, it is to be subtracted.

It has already been observed, that that part of the arch beyond 0 towards the right-hand is called the arch of excess: the nonius, when the 0 on it is at that part, must be read the contrary way, or, which is the same thing, you may read off the minutes in the usual way, and then their complement to 20 minutes will be the real number to be added to the degrees and minutes pointed out by the 0 on the nonius.

III. *To set the fore horizon glass perpendicular to the plane of the quadrant.*

Having previously made the above adjustment, incline the quadrant on one side as much as possible, provided the horizon continues to be seen in both parts of the glass; if, when the

instrument is thus inclined, the edge of the sea seen through the lower hole of the sight vane continues to form one unbroken line, the horizon glass is perfectly adjusted; but if the reflected horizon be separated from that seen by direct vision, the speculum is not perpendicular to the plane of the quadrant: then if the limb of the quadrant is inclined towards the horizon, with the face of the instrument upwards, and the reflected sea appears higher than the real sea, you must slacken the screw before the horizon glass, and tighten that which is behind it; but if the reflected sea appears lower, the contrary must be performed. Care must be always taken in this adjustment to loosen one screw before the other is screwed up, and to leave the adjusting screws tight, so as to draw with a moderate force against each other.

This adjustment may be also made by the sun, moon, or a star: in this case the quadrant is to be held in a vertical position; if the image seen by reflection appears to the right or left of the object seen directly, then the glass must be adjusted as before by the two screws.

It will be necessary, after having made this adjustment, to examine if the horizon glass still continues to be parallel to the index glass, as sometimes by turning the sunk screws the plane of the horizon glass will have its position altered.

USE OF HADLEY'S QUADRANT.

The use of the quadrant is to ascertain the angle subtended by two distant objects at the eye of the observer; but principally to observe the altitude of a celestial object above the horizon. This is pointed out by the index when one of the objects seen by reflection is made to coincide with the other, seen through the transparent part of the horizon glass.

To take an altitude of the sun, moon, or a star, by a fore observation.

Having previously adjusted the instrument, place the 0 on the nonius opposite to 0 on the arch, and turn down one or more of the screens, according to the brightness of the sun; then apply the eye to the upper hole in the fore sight vane, if the sun's image be very bright, otherwise to the lower, and holding the quadrant vertically, look directly towards the sun, so as to let it be behind the silvered part of the horizon glass, then the coloured sun's image will appear on the speculum; move the index forward till the sun's image, which will appear to descend, just touches the horizon with its lower or upper limb; if the upper hole be looked through, the sun's image must be made to appear in the middle of the trans arent part

of the horizon, but if it be the lower hole, hold the quadrant so that the sun's image may be bisected by the line joining the silvered and transparent parts of the horizon glass.

The sun's limb ought to touch that part of the horizon immediately under the sun, but as this point cannot be exactly ascertained, it will be therefore necessary for the observer to give the quadrant a slow motion from side to side, turning at the same time upon his heel, by which motion the sun will appear to sweep the horizon, and must be made just to touch it at the lowest part of the arch; the degrees and minutes then pointed out by the index on the limb of the quadrant will be the observed altitude of that limb which is brought in contact with the horizon.

When the meridian or greatest altitude is required, the observation should be commenced a short time before the object comes to the meridian; being brought down to the horizon, it will appear for a few minutes to rise slowly; when it is again to be made to coincide with the horizon by moving the index forward; this must be repeated until the object begins to descend, when the index is to be secured, and the observation to be read off.

From this description of the quadrant and its use, the manner of adjusting and using the sextant will be readily apprehended. Our limits will not allow a particular description of this excellent instrument.

The Artificial Horizon.

In many cases it happens that altitudes are to be taken on land by the quadrant or sextant; which, for want of a natural horizon, can only be obtained by an artificial one. There has been a variety of these sorts of instruments made, but the kind now described is allowed to be the only one that can be depended upon. It consists of a wooden or metal-framed roof, containing two true parallel glasses of about 5 by $2\frac{1}{2}$ inches, fixed not too tight in the frames of the roof. This serves to shelter from the air a wooden trough filled with quicksilver. In making an observation by it with the quadrant or sextant, the reflected image of the sun, moon, or other object is brought to coincide with the same object reflected from the glasses of the quadrant or sextant: half the angle shown upon the limb is the altitude above the horizon or level required. It is necessary in a set of observations that the roof be always placed the same way. When done with, the roof folds up flatwise, and, with the quicksilver in a bottle, &c. is packed into a portable flat case.

SECTION III.

VARIATION OF THE COMPASS.

The variation of the compass is the deviation of the points of the mariner's compass from the corresponding points of the horizon, and is termed east or west variation according as the magnetic needle or north point of the compass is inclined to the eastward or westward of the true north point of the horizon.

The true amplitude of any celestial object is an arch of the horizon contained between the true east or west points thereof and the centre of the object at the time of its rising or setting; or it is the degrees and minutes the object rises or sets to the northward or southward of the true east or west points of the horizon.

The magnetic amplitude is an arch contained between the east or west points of the compass and the centre of the object at rising or setting; or it is the bearing of the object by compass when in the horizon.

The true azimuth of an object is an arch of the horizon contained between the true meridian and the azimuth circle passing through the centre of the object.

The magnetic azimuth is an arch contained between the magnetic meridian and the azimuth circle passing through the centre of the object; or it is the bearing of the object by compass at any time when it is above the horizon.

The true amplitude or azimuth is found by calculation, and the magnetic amplitude or azimuth by an azimuth compass.

The magnetic amplitude or azimuth of the sun, or any celestial object, may be accurately observed by Mr. M'Culloch's patent compass, of which the following is a description.

DESCRIPTION OF THE AZIMUTH COMPASS.

Frontispiece, fig. 4, contains a perspective view of the azimuth compass ready for observation. The needle and card of this compass are similar to those of the steering compass, with this difference only, that a circular ring of silvered brass, divided into 360°, or rather four times 90°, circumscribes the card: *b* represents the compass-box, which is of brass, and has a hollow conical bottom; *e* is the prop or support of the compass-box, which stands in a brass socket screwed to the bottom of the wooden box, and may be turned round at pleasure; *h* is one of the guards, the other, being directly opposite, is hid by the box,—each guard has a slit, in which a pin projecting from the side of the box may move freely in a vertical direction; 1 is a brass bar, upon which, at right angles, the side-vanes are fixed,—a line is drawn along the middle of this bar, which line, the lines in the vanes, and the threads joining their tops are in

the same plane; 2 is a coloured glass moveable in the vane 3; 4 is a magnifying glass moveable in the other vane, whose focal distance is nearly equal to the distance between the vanes; 5 is the vernier, which contains six divisions, and as the limb of the card is divided into half-degrees, each division of the vernier is therefore five minutes,—the interior surface of the vernier is ground to a sphere, whose radius is equal to that of the card; 6 is a slide or stopper connected with the vernier, which serves to push the vernier close to the card, and thereby prevent it from vibrating as soon as the observation of the amplitude or azimuth is completed, and hence the degrees and parts of a degree may be read off at leisure with certainty; 7 is a convex glass, to assist the eye in reading off the observed amplitude or azimuth.

To observe the sun's amplitude.

Turn the compass-box until the vane containing the magnifying glass is directed towards the sun; and when the bright speck, or rays of the sun collected by the magnifying glass, falls upon the slit in the other vane, stop the card by means of the nonius, and read off the amplitude.

Without using the magnifying glass, the sight may be directed through the dark glass towards the sun; and in this case the card is to be stopped when the sun is bisected by the thread in the other vane.

The observation should be made when the sun's lower limb appears somewhat more than his semidiameter above the horizon, because his centre is really then in the horizon, although it is apparently elevated on account of the refraction of the atmosphere: this is particularly to be noticed in high latitudes.

To observe the sun's azimuth.

Raise the magnifying glass to the upper part of the vane, and move the box, as before directed, until the bright speck fall on the other vane or on the line in the horizontal bar; the card is then to be stopped, and the divisions being read off will be the sun's magnetic azimuth.

If the card vibrate considerably at the time of observation, it will be better to observe the extreme vibrations and take their mean as the magnetic azimuth. When the magnetic azimuth is observed, the altitude of the object must be taken in order to obtain the true azimuth.

It will conduce much to accuracy if several azimuths be observed, with the corresponding altitudes, and the mean of the whole taken for the observation.

To find the variation of the compass by an amplitude.

RULE.—1. To the log. secant of the latitude, rejecting the index, add the log. sine of the sun's declination, corrected for the time and place of observation; their sum will be the log. sine of the true amplitude, to be reckoned from the east in the

morning or the west in the afternoon, towards the north or south, according to the declination.

2. Then if the true and magnetic amplitudes be both north or both south their difference is the variation, but if one be north and the other south their sum is the variation; and to know whether it be easterly or westerly, suppose the observer looking towards that point of the compass representing the magnetic amplitude; then if the true amplitude be to the right-hand of the magnetic amplitude the variation is east, but if to the left-hand it is west.

EXAMPLE I.

July 3, 1812, in latitude 9° 36′ S. the sun was observed to rise E. 12° 42′ N.; required the variation of the compass.

Latitude	9°	36′ S.	- -	Secant	0.00613
Declination	22	59 N.	- -	Sine	9.59158
True amplitude E.	23	20 N.	- -	Sine	9.59771
Mag. amplitude E.	12	42 N.			
Variation	10	38 W., because the true amplitude is to the left of the magnetic.			

EXAMPLE II.

September 24, 1812, in latitude 26° 32′ N. and longitude 78° W. the sun's centre was observed to set W. 6° 15′ S. about 6h. P. M.; required the variation of the compass.

Sun's declination	0°	30′ S.			
Corr. for long. 78° W.	+	5			
Corr. for time 6h. P. M.	+	6			
Reduced declination	0	41	- - -	Sine	8.07650
Latitude	26	32	- - -	Secant	0.04834
True amplitude W.	0	46 S.	- -	Sine	8.12484
Mag. amplitude W.	6	15 S.			
Variation	5	29 E., because the true amplitude is to the right-hand of the magnetic.			

To find the variation of the compass by an azimuth.

RULE.—1. Reduce the sun's declination to the time and place of observation, and compute the true altitude of the sun's centre.

2. Subtract the sun's declination from 90° when the latitude and declination are of the same name, or add it to 90° when they are of contrary names, and the sum or remainder will be the sun's polar distance.

3. Add together the sun's polar distance, the latitude of the place, and the altitude of the sun; take the difference between half their sum and the polar distance, and note the remainder.

4. Then add together
the log. secant of the altitude
the log. secant of the latitude } rejecting their indices,
the log. co-sine of the half-sum,
and the log. co-sine of the remainder.

5. Half the sum of these four logarithms will be the sine of an arch, which doubled will be the sun's true azimuth; to be reckoned from the south in north latitude, and from the north in south latitude; towards the east in the morning, and towards the west in the afternoon.

6. Then if the true and observed azimuths be both on the east or both on the west side of the meridian, their difference is the variation; but if one be on the east and the other on the west side of the meridian, their sum is the variation: and to know if it be east or west, suppose the observer looking towards that point of the compass representing the magnetic azimuth; then if the true azimuth be to the right of the magnetic, the variation is east, but if the true be to the left of the magnetic the variation is west.

EXAMPLE.

November 2, 1812, in latitude 25° 32′ N. and longitude 75° W. the altitude of the sun's lower limb was observed to be 15° 36′, about 4h. 10m. P. M., his magnetic azimuth at that time being S. 58° 32′ W., and the height of the eye 18 feet; required the variation of the compass.

Sun's dec. Nov. 2, at n.	14°	48′ S.	Obs. alt. sun's lower limb		15° 36′
Corr. for long. 75° W.	+	4	Semidiameter 16′ }		+ 12
Co. for ti. 4h. 10m. af. n.	+	3	Dip 4 }		
Reduced declination	14	55			15 48
	90	00	Refraction		3
Polar distance	104	55	True altitude		15 45
Altitude	15	45	- - Secant	0.01662	
Latitude	25	32	- - Secant	0.04463	
Sum	146	12			
Half	73	6	- - Co-sine	9.46345	
Remainder	31	49	- - Co-sine	9.92929	
				19.45399	
	32	14	- - Sine	9.72699	
		2			
True azimuth S.	64	28 W.			
Mag. azimuth S.	58	32 W.			
Variation	5	56 east, because the true azimuth is to the right of the magnetic.			

To draw a true meridian line to a map, having the variation and magnetical meridian given.

On any magnetical meridian or parallel, upon which the map is protracted, set off an angle from the north towards the east, equal to the degrees or quantity of variation if it be westerly, or from the north towards the west if it be easterly, and the line which constitutes such an angle with the magnetical meridian will be a true meridian line.

For if the variation be westerly, the magnetical meridian will be the quantity of variation of the west side of the true meridian, but if easterly, on the east side; therefore the true meridian must be a like quantity on the east side of the magnetical one when the variation is westerly, and on the west side when it is easterly.

To lay out a true meridian line by the circumferentor.

If the variation be westerly, turn the box about till the north of the needle points as many degrees from the flower-de-luce towards the east of the box, or till the south of the needle points the like number of degrees from the south towards the west, as are the number of degrees contained in the variation, and the index will be then due north and south; therefore, if a line be struck out in the direction thereof, it will be a true meridian line.

If the variation was easterly, let the north of the needle point as many degrees from the flower-de-luce towards the west of the box, or let the south of the needle point as many degrees towards the east, as are the number of degrees contained in the variation, and then the north and south of the box will coincide with the north and south points of the horizon, and consequently a line being laid out by the direction of the index will be a true meridian line.

This will be found to be very useful in setting a horizontal dial, for if you lay the edge of the index by the base of the stile of the dial, and keep the angular point of the stile towards the south of the box, and allow the variation as before, the dial will then be due north and south, and in its proper situation, provided the plane upon which it is fixed be duly horizontal, and the sun be south at noon; but in places where it is north at noon the angular point of the index must be turned to the north.

How maps may be traced by the help of a true meridian line.

If all maps had a true meridian line laid out upon them, it would be easy, by producing it, and drawing parallels, to make out field-notes; and by knowing the variation, and allowing it upon every bearing, and having the distances, you would have notes sufficient for a trace. But a true meridian line is seldom to be met with; therefore we are obliged to have recourse to the foregoing method. It is therefore advised to lay out a true meridian line upon every map.

To find the difference between the present variation, and that at a time when a tract was formerly surveyed, in order to trace or run out the original lines.

If the old variation be specified in the map or writings, and the present be known, by calculation or otherwise, then the difference is immediately seen by inspection; but as it more frequently happens that neither is certainly known, and as the variation of different instruments is not always alike at the same time, the following practical method will be found to answer every purpose.

Go to any part of the premises where any two adjacent corners are known; and if one can be seen from the other, take their bearing; which, compared with that of the same line in the former survey, shows the difference. But if trees, hills, &c. obstruct the view of the object, run the line according to the given bearing, and observe the nearest distance between the line so run and the corner, then,

As the length of the whole line
Is to 57.3 degrees,*
So is the said distance
To the difference of variation required.

EXAMPLE.

Suppose it be required to run a line which some years ago bore NE. 45°, distance 80 perches, and in running this line by the given bearing, the corner is found 20 links to the left-hand; what allowance must be made on each bearing to trace the old lines, and what is the present bearing of this particular line by the compass?

P.		Deg.		L.
As 80	:	57.3	: :	20
25		20		
2|000		1146.0(0° 34′		
		60		
		2)68|760.0		

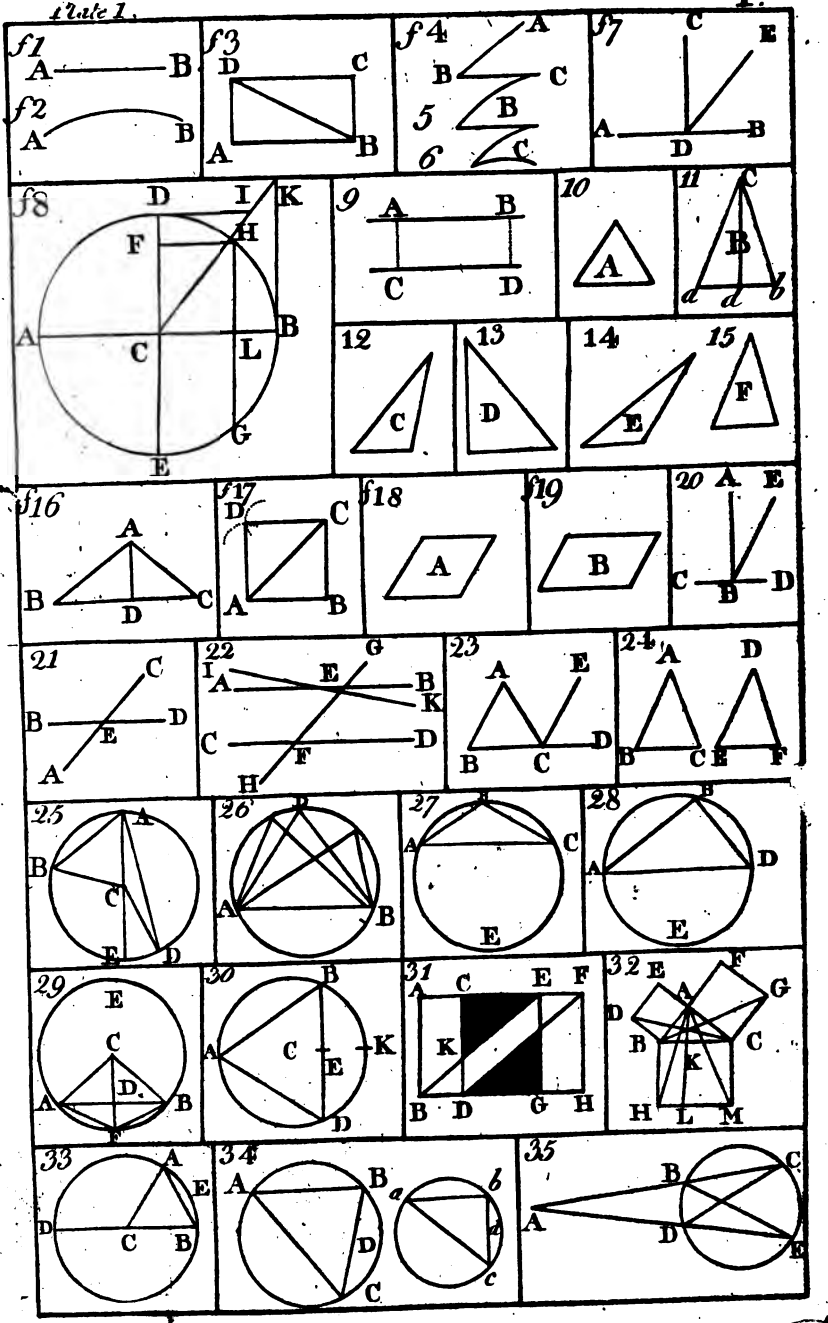
Answer, 34 minutes, or a little better than half a degree to the left-hand, is the allowance required, and the line in question bears N. 44° 26′ E.

Note.—The different variations do not affect the area in the calculation, as they are similar in every part of the survey.

* 57.3 is the radius of a circle (nearly) in such parts as the circumference contains 360.

THE END

Plate I.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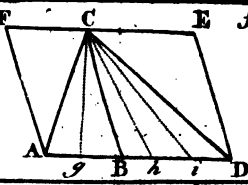

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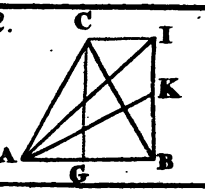

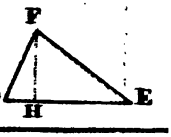

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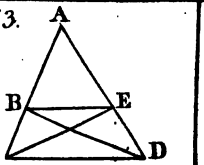

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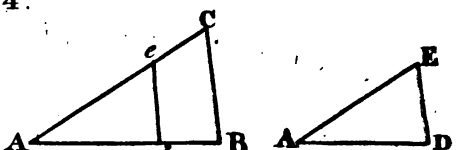

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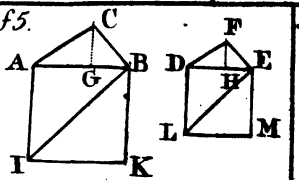

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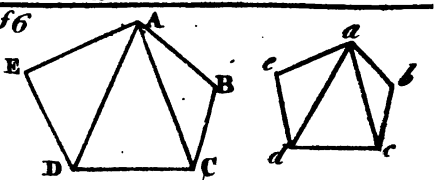

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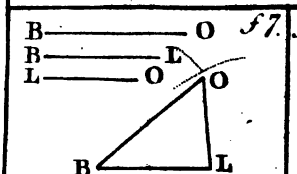

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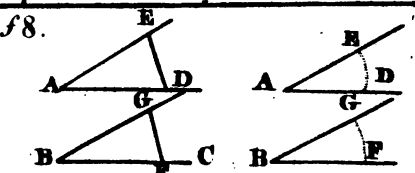

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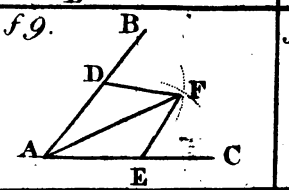

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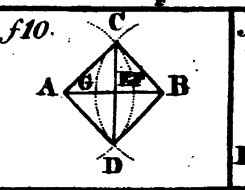

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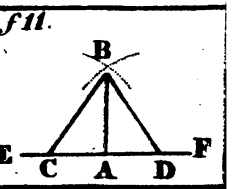

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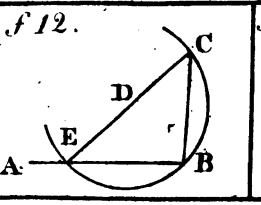

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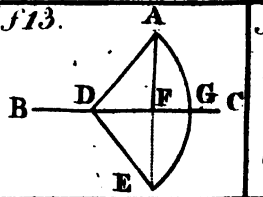

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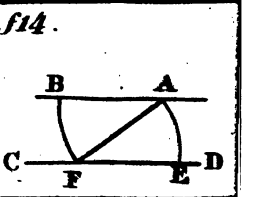

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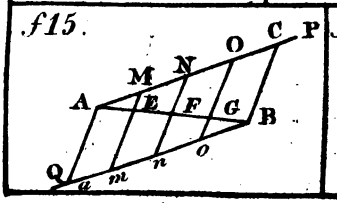

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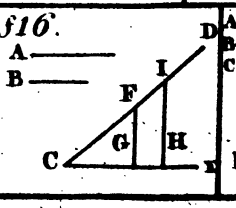

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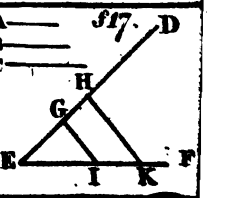

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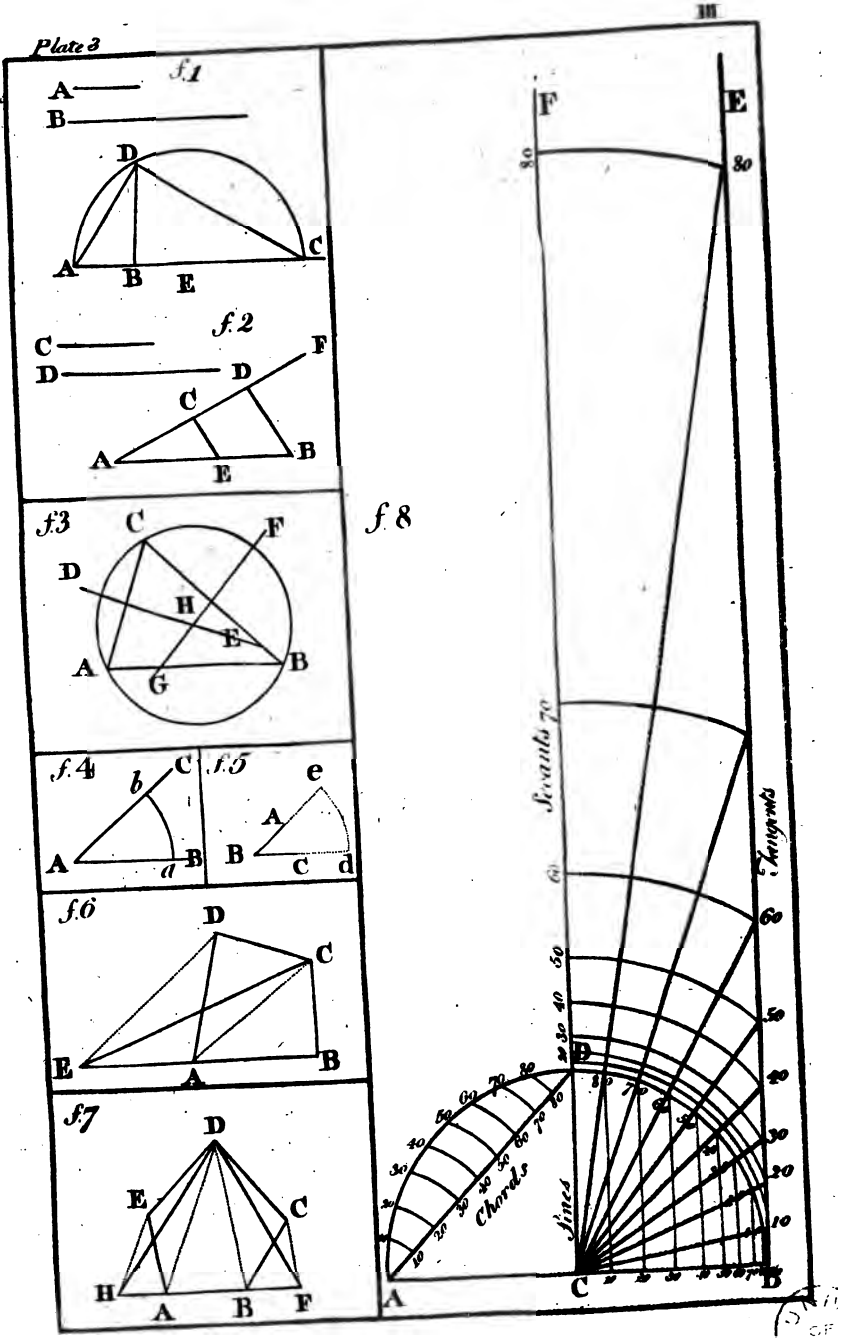
Plate 3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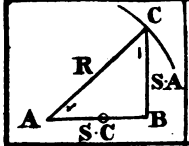
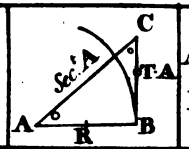
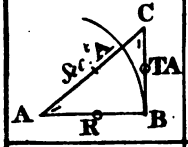
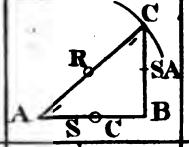
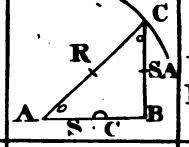
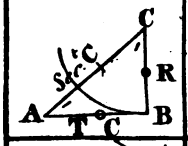
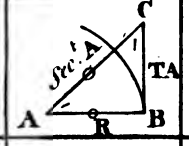
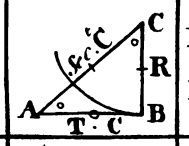
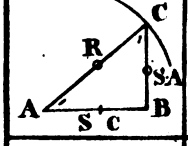
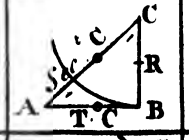
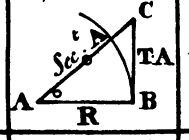
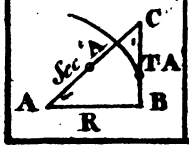
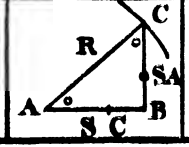
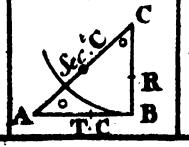
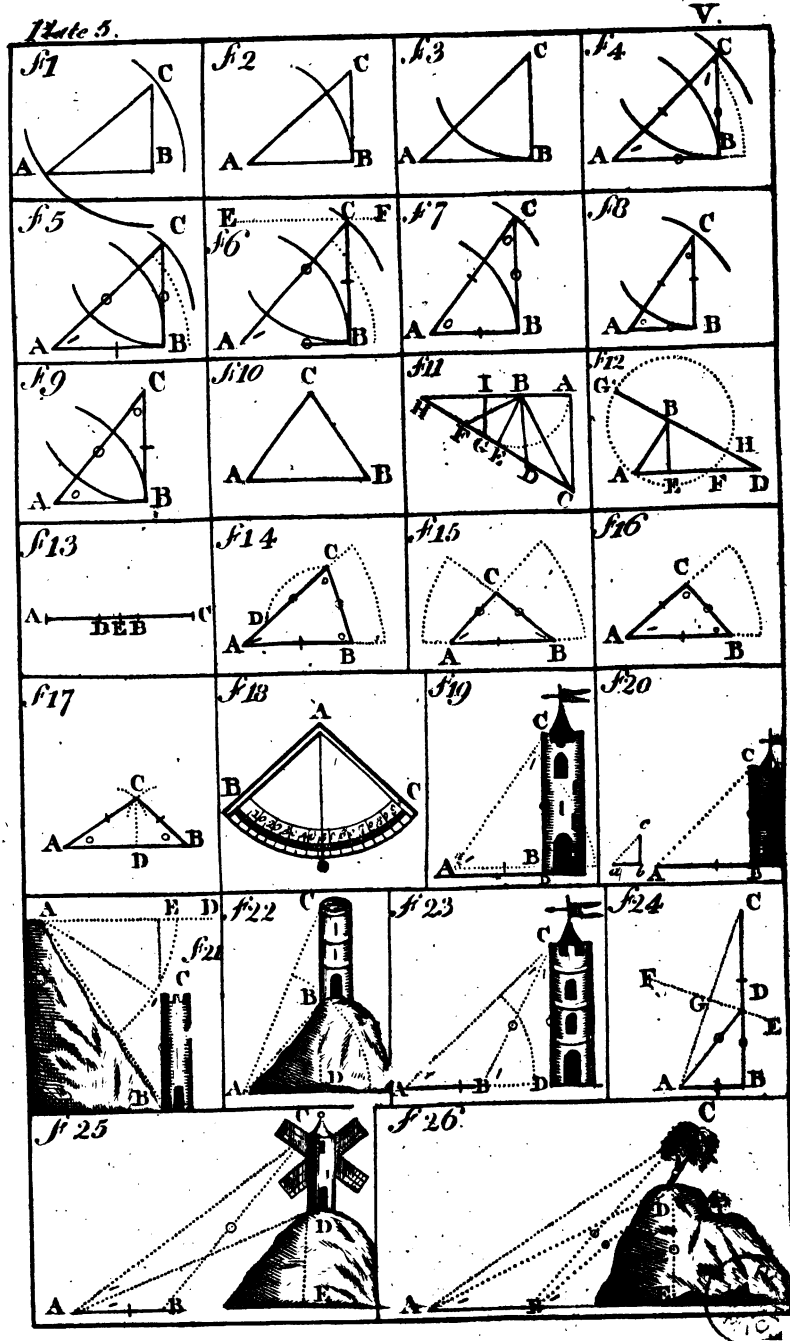
The Proportions for the Solution of 6 Cases of Right Angled Plain Triangles

	*Case 1.* R : *ac* :: SA : BC R : *ac* :: SC : AB		TC : AB :: R : BC. TC : AB :: *Sec.* C : AC		AB : R :: AC : *Sec.* A R : AB :: TA : BC
	*Sec.* A : AC :: R : AB *Sec.* A : AC :: TA : BC		*Case 3.* SA : BC :: R : AC SA : BC :: SC : AB		*Case 5.* AC : R :: BC : SA R : AC :: SC : AB
	*Sec.* C : AC :: R : BC *Sec.* C : AC :: TC : AB		TA : BC :: R : AB TA : BC :: *Sec.* A : AC		BC : R :: AC : *Sec.* C R : BC :: TC : AB
	*Case 2.* SC : AB :: R : AC SC : AB :: SA : BC		R : BC :: TC : AB R : BC :: *Sec.* C : AC		*Case 6.* AB : R :: BC : TA R : AB :: *Sec.* A : AC
	R : AB :: TA : BC R : AB :: *Sec.* A : AC		*Case 4.* AC : R :: AB : SC R : AC :: SA : BC		BC : R :: AB : TC R : BC :: *Sec.* C : AC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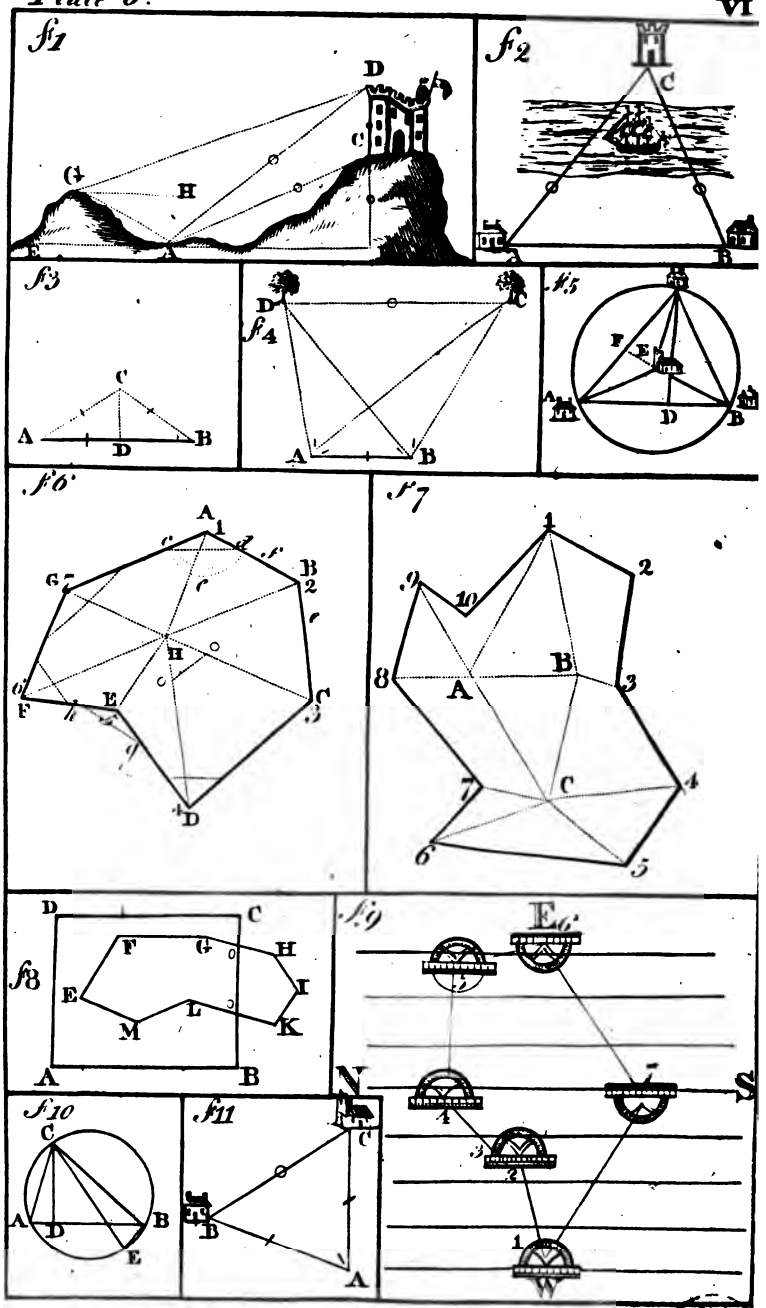
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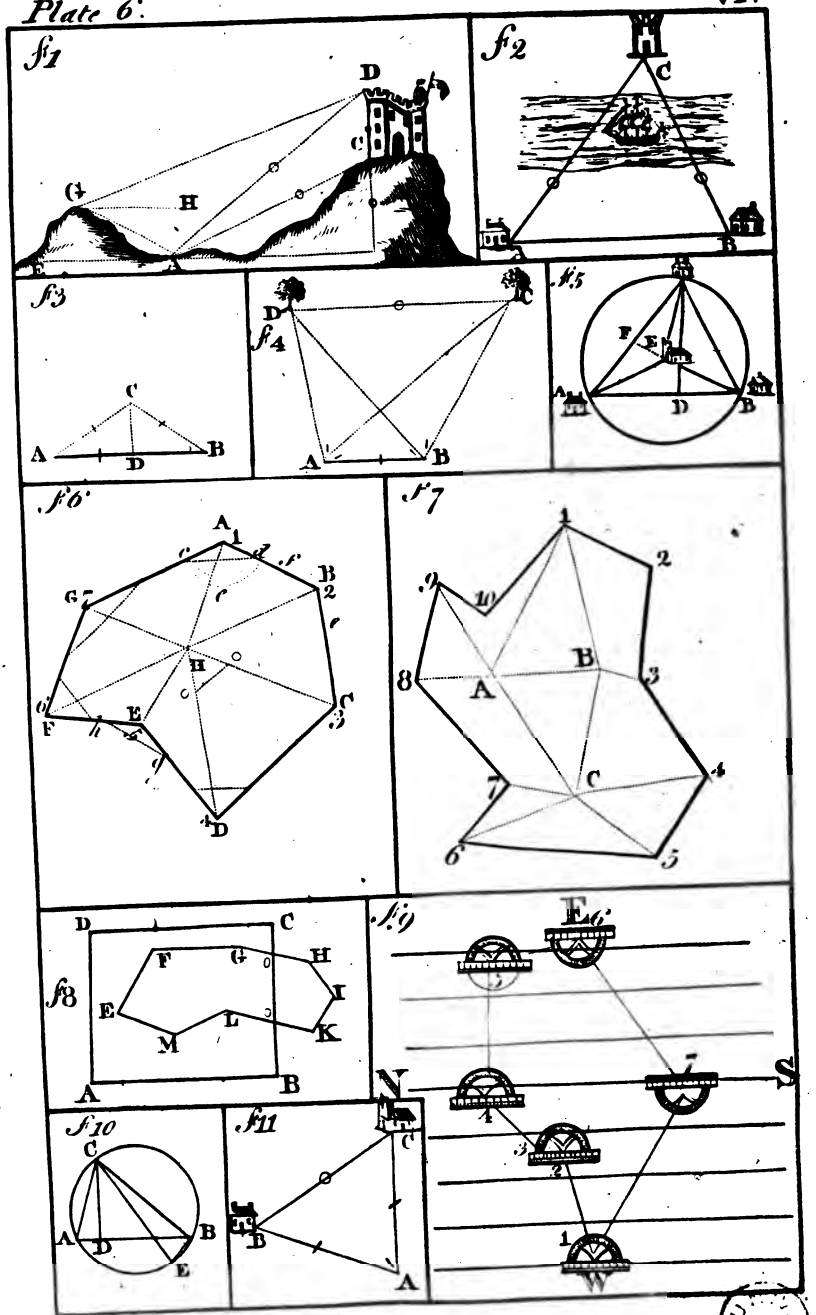
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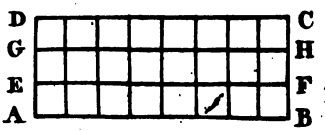

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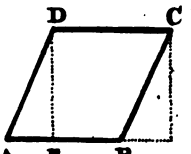

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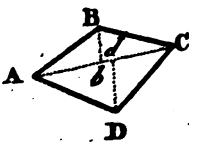

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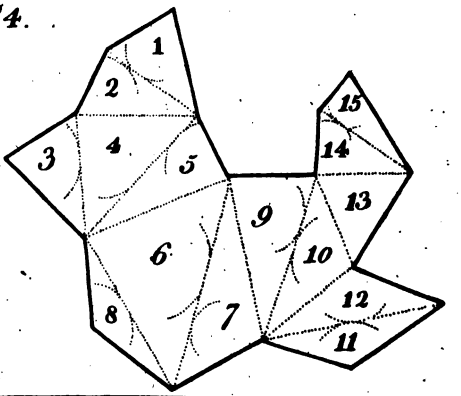

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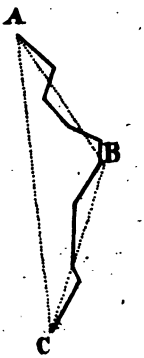

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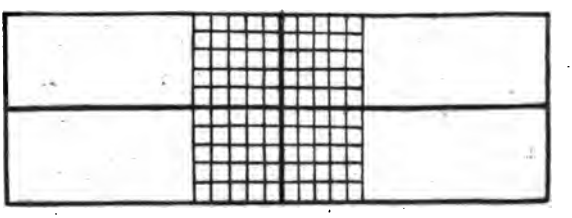

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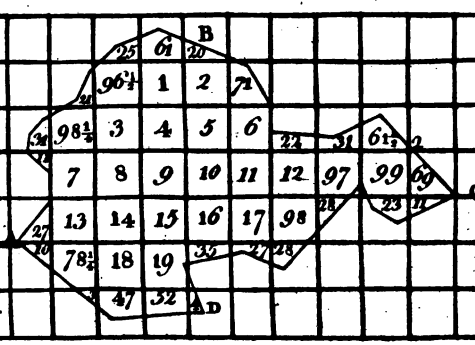

Plate 8.

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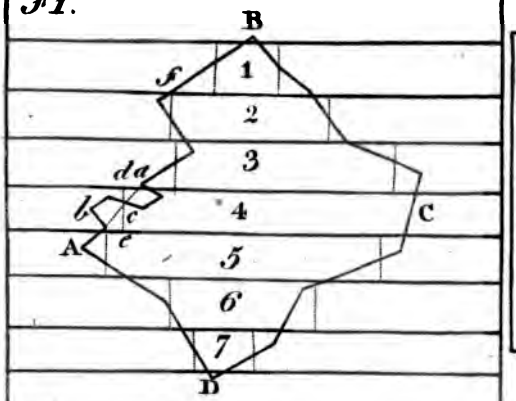

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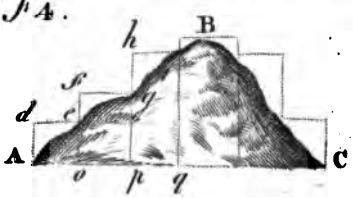

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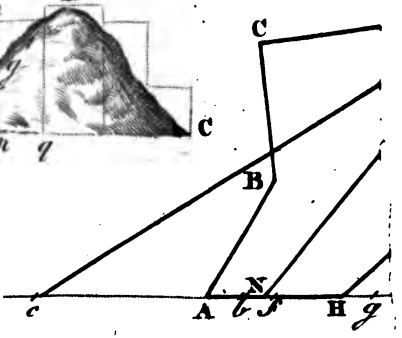

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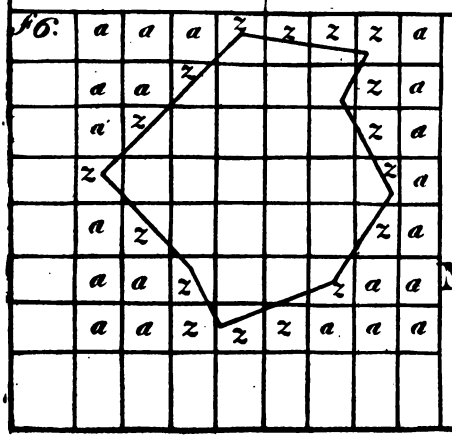

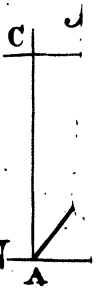

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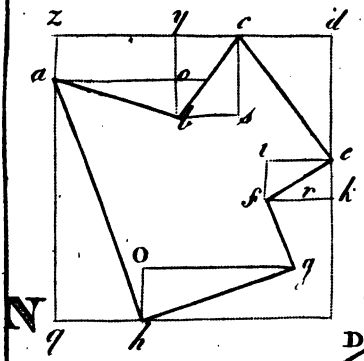
Fig 1.



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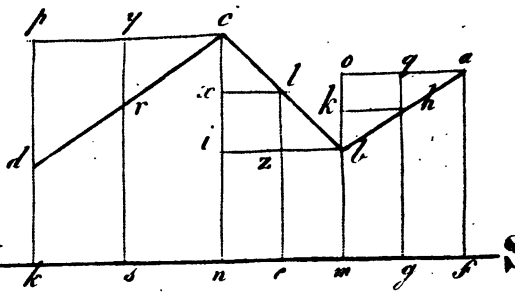

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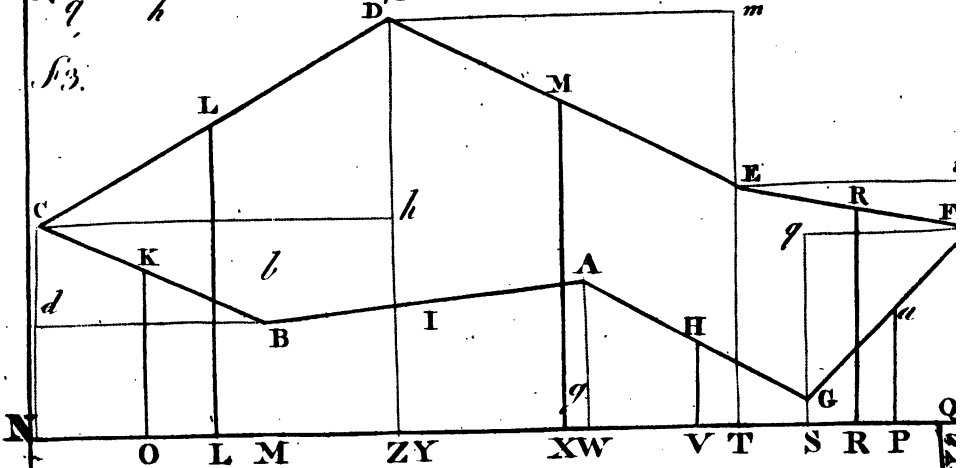

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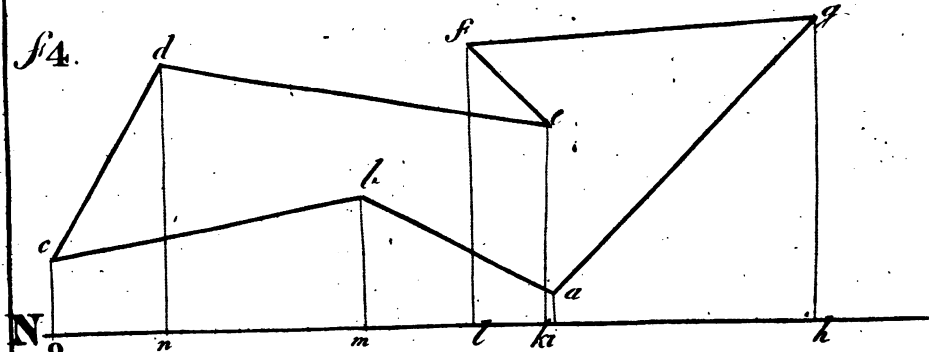

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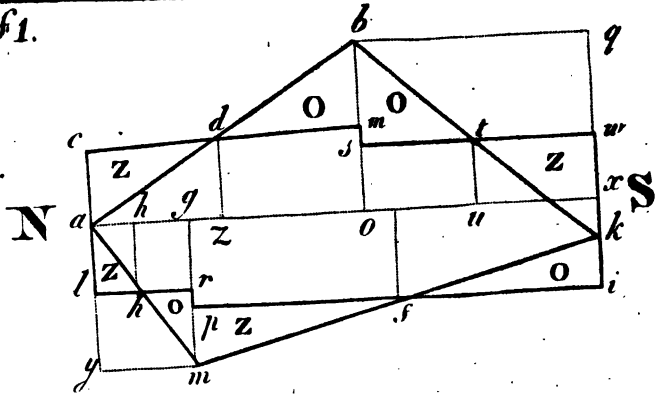
Fig 1.



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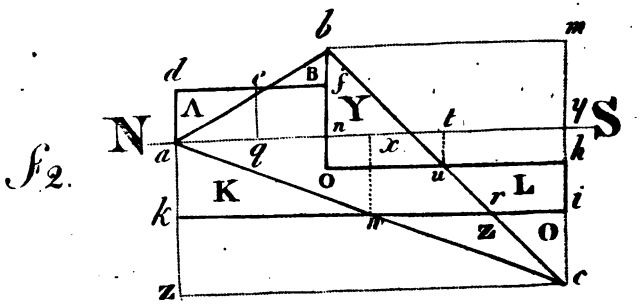
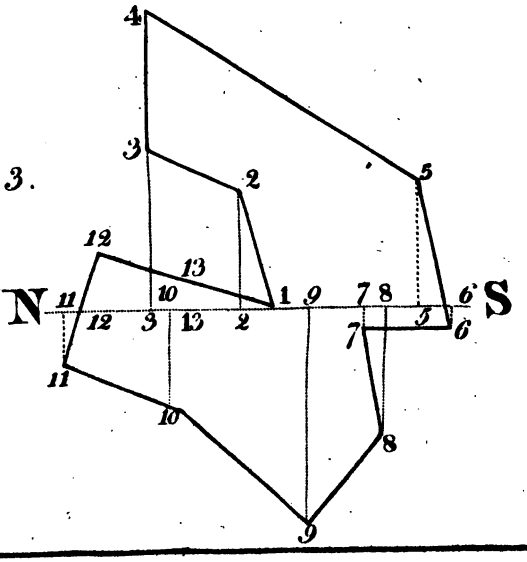


Fig 3.



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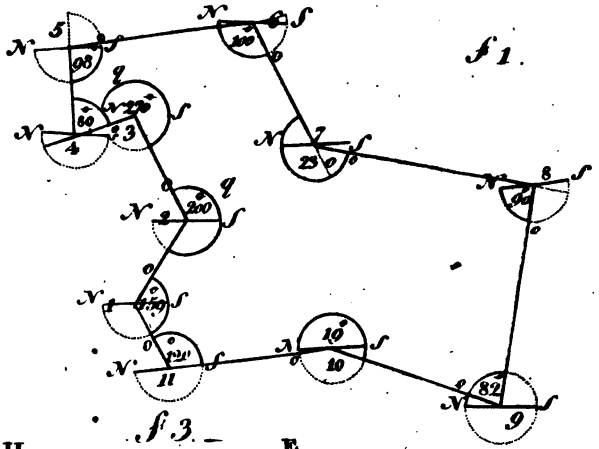
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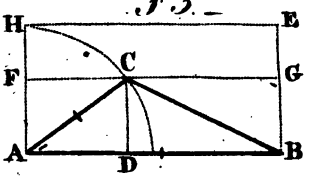
fig 3.
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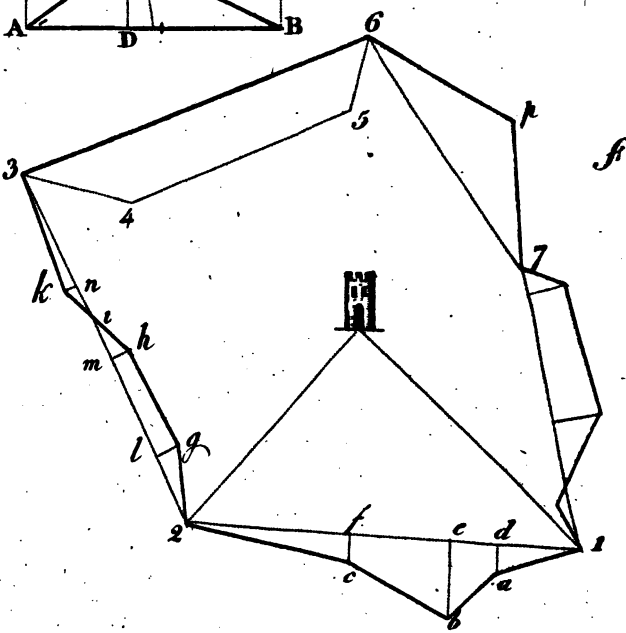
fig 2.
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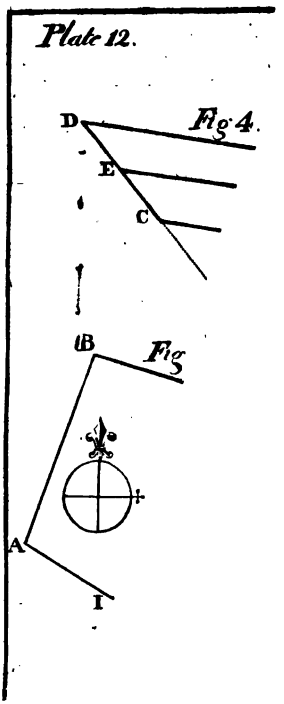
Plate 12.



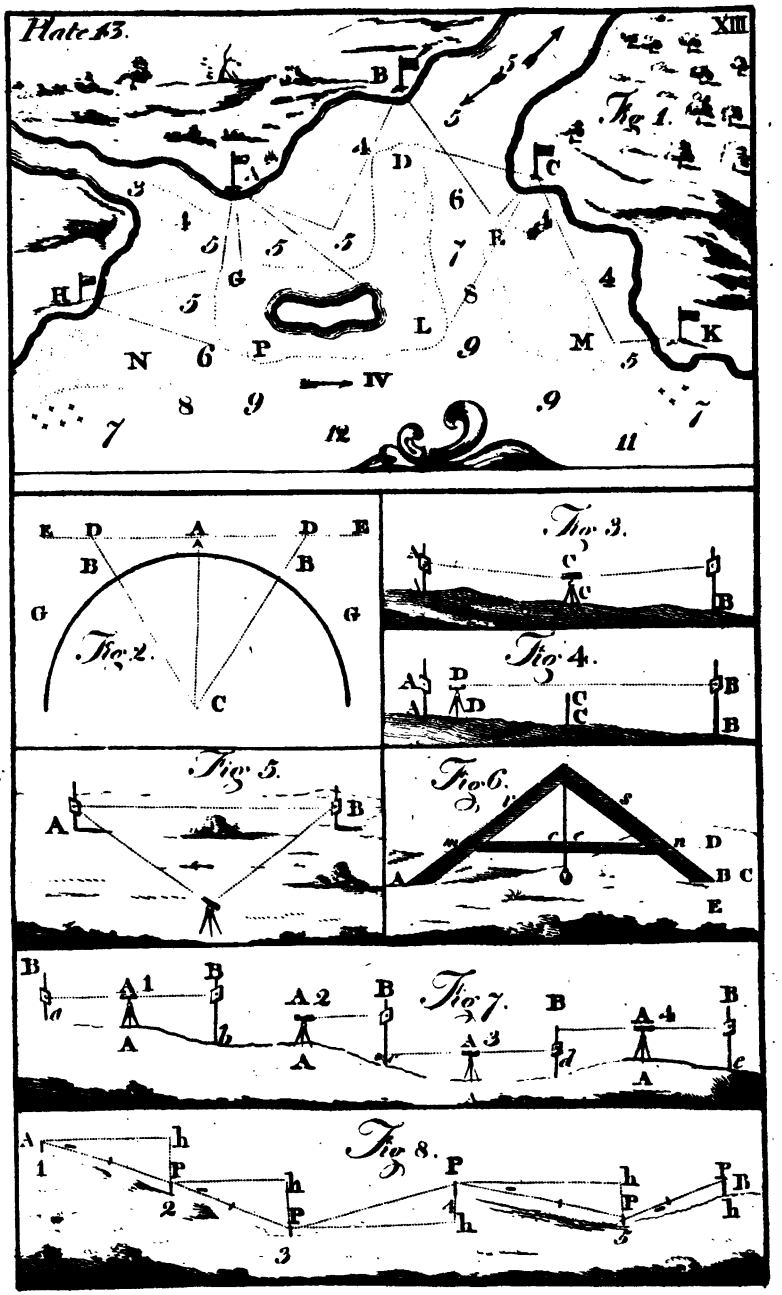


Plate 43.
XIII
Fig. 1.
Fig. 2.
Fig. 3.
Fig. 4.
Fig. 5.
Fig. 6.
Fig. 7.
Fig. 8.